DIOPHANTINE TRIPLES AND THE PTOLEMY RELATION

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Abstract. We discuss the construction of Diophantine triples from the Farey diagram and the action of the modular group on the set of such triples.

This is a collaboration with GitHub Copilot and ChatGPT.

1. Introduction

A *Diophantine triple* is a set of three distinct positive integers $A = \{a, b, c\}$ such that the product of any two integers from the set, increased by 1, results in a perfect square. In other words, for a Diophantine triple $\{a, b, c\}$, the following conditions must hold:

$$ab + 1 = \text{perfect square} \tag{1}$$
$$ac + 1 = \text{perfect square} \tag{2}$$
$$bc + 1 = \text{perfect square} \tag{3}$$

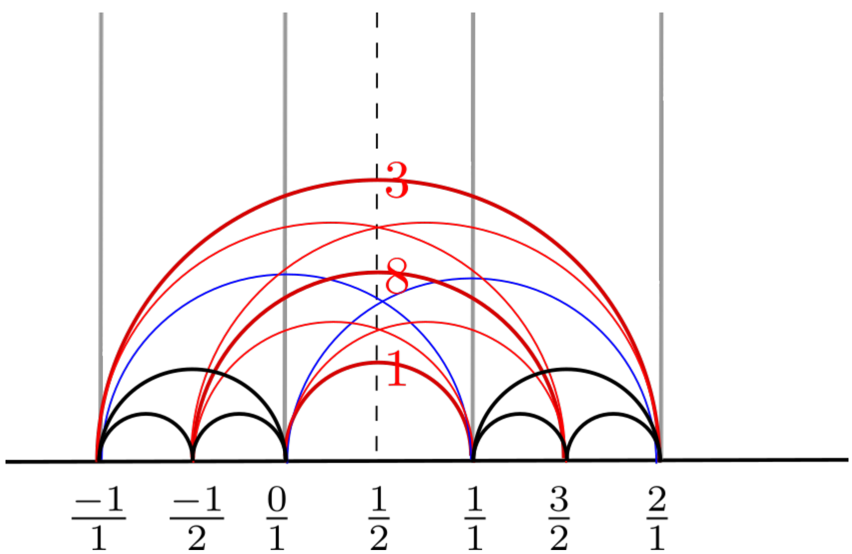

Figure 1. A pair of ideal triangles from the farey diagram that are swapped by the involution $z \mapsto 1 - \bar{z}$. The λ-length of the three central semicircles invariant under this involution are respectively $3, 8$ and 1.

For example, one well-known Diophantine triple is $\{1, 3, 8\}$, because:

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$$1 + 3 \times 8 = 5^2 \tag{4}$$

$$1 + 3 \times 1 = 2^2 \tag{5}$$

$$1 + 8 \times 1 = 3^2 \tag{6}$$

There are also Diophantine quadruples and higher sets. A Diophantine set is a set of positive integers A with the property that the product of any two distinct elements of A increased by 1 is a perfect square There is a vast literature, dating back to Diophantus of Alexandria see the survey by Dujella [4] for an account. The most important result states that such sets A can have at most five elements, and there are only finitely many of them with five elements [2].

In this note we give a geometric construction of Diophantine triples using the Farey diagram and the notion of λ-length due to Penner [9]. The λ-length of each edge of an ideal triangle in the Farey diagram is 1. We begin with a pair of ideal triangles in the Farey diagram which are swapped by the involution $z \mapsto 1 - \bar{z}$. The relations (1), (2) and (3) arise naturally from the Ptolemy relation for an ideal quadrilateral in the Farey diagram.

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Some of the text of this paper was suggested by GitHub Copilot[13, 14] and ChatGPT.

And finally I thank the referee for their comments and suggestions.

2. Geometry of the Farey diagram

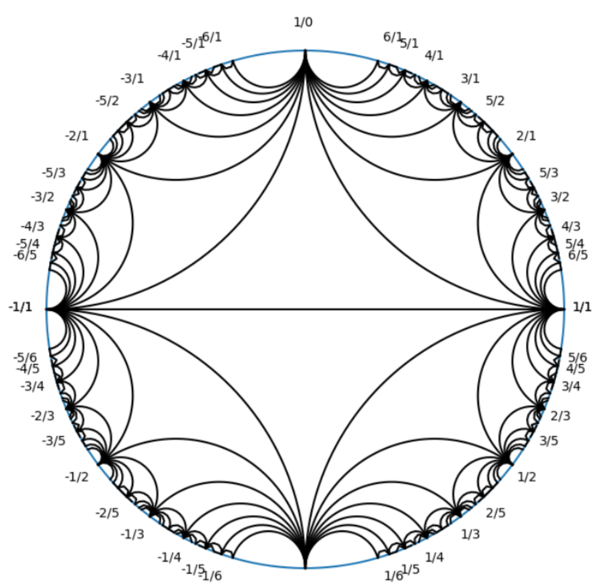

Figure 2. Farey diagram.

2.1. Farey diagram and λ-lengths. The Farey diagram (see Figure 2) is a geometric representation of the relationships between rational numbers, closely associated with the Farey sequence and the Stern–Brocot tree. The Farey diagram is often visualized as a set of points on the unit circle or as a triangulation of the hyperbolic plane. The diagram has applications in number theory [10], hyperbolic geometry, and the study of continuedfractions [11, 12].

It provides a visual and geometric interpretation of how rational numbers are related to each other, particularly their mediants and properties of the modular group.

2.2. Construction. The Farey diagram can be constructed by representing rational numbers as points on the real line viewed as the ideal boundary of the Poincaré half plane $\mathbb{H}$. Pairs of rational numbers $\frac{a}{c}, \frac{b}{d}$ are joined by a Poincaré geodesic (often referred to as an *arc*) if the determinant of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is ± 1 and such a pair are called *Farey neighbors*. The mediant of two fractions $\frac{a}{b}$ and $\frac{c}{d}$ is given by:

$$\text{mediant}\left(\frac{a}{c}, \frac{b}{d}\right) = \frac{a+b}{c+d}$$

Note that the mediant of a pair of Farey neighbors is a Farey neighbor of each of them. A triple of Farey neighbors form the set of vertices of an ideal triangle in the Farey diagram: for example the triple $1 = 1/1, 2 = 2/1, 3/2$ in Figure 1 is an ideal triangle in the Farey diagram. More generally, if $b/d < a/c$ are Farey neighbors then the triple $b/d < (a+b)/(c+d) < a/c$ are the vertices of an ideal triangle in the Farey diagram.

2.3. Symmetries. The Farey diagram is invariant under the action of the modular group

$$\Gamma = \mathrm{PSL}(2,\mathbb{Z}) < \mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I\}.$$

An element of $\mathrm{PSL}(2,\mathbb{Z})$ acts on the Farey diagram by fractional linear transformations that is:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

The action is transitive on the set of Farey neighbors since the image of $\infty = 1/0$ is a/c and the image of $0 = 0/1$ is b/d. The Farey diagram is also invariant under the orientation reversing involutions

$$\begin{aligned} z &\mapsto -\bar{z} \\ z &\mapsto 1-\bar{z}. \end{aligned}$$

Note that these involutions are not conjugate by an element of $\mathrm{PSL}(2,\mathbb{Z})$.

2.4. λ-lengths. Penner introduced the notion of λ-length to study the geometry of the decorated Teichmüller space of a punctured surface [9]. We give an equivalent formulation for arcs in the half plane $\mathbb{H}$ joining pairs of extended rationals $\frac{a}{c}, \frac{b}{d}$ and define the λ-length of an arc to be the absolute value of the determinant of the matrix:

$$\begin{pmatrix} a & b \\ c & d. \end{pmatrix}$$

Although we will not use it here, we note that this has a geometric interpretation: the λ-length is the exponential of half the length of the portion of the arc outside the Ford circles tangent at $a/c, b/d$.

The *Ptolemy relation* is a classical result from Euclidean geometry that relates the lengths of the sides and diagonals of a cyclic quadrilateral (a quadrilateral inscribed in a circle). It states that for any cyclic quadrilateral, the sum of the products of its two pairs of opposite sides is equal to the product of its diagonals. The relation provides an important bridge between geometry and algebraic structures, particularly when studying configurations of points and lengths. Penner proved a version of the Ptolemy relation for the λ-lengths of the sides of an ideal quadrilateral.

Lemma 2.1. If A, A' and B, B' denote the λ-lengths of opposite sides and D, D' the λ-lengths of the diagonals of an ideal of an ideal quadrilateral then the Ptolemy relation is:

$$A.A' + B.B' = D.D'.$$

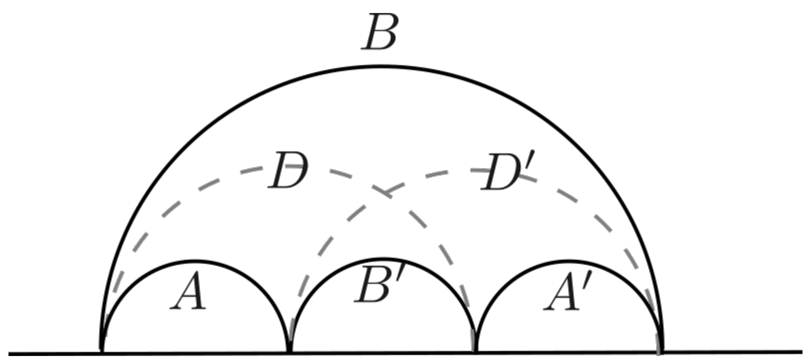

FIGURE 3. The Ptolemy relation for an ideal quadrilateral.

To illustrate this consider one of the ideal quadrilaterals in Figure 1, the quadrilateral with vertices $-1, -1/2, 3/2, 2$ say, then the λ-lengths of the sides are $B = 8, B' = 3$ and $A = A' = 1$ the diagonals have the same length $D = D' = 5$ and so the Ptolemy relation gives:

$$1 \times 1 + 3 \times 8 = 5 \times 5,$$

which is (4) above.

2.5. **Main result.**

Theorem 2.2. If $a/c < (a+b)/(c+d) < b/d$ are a triple of Farey neighbors then

$$(2b-d)d, (2a-c)c, (2(a+b)-c-d)(c+d)$$

is a Diophantine triple.

This can be checked by direct calculation. For example for the pair $(2b-d)d, (2a-c)c$ Figure 4 have:

$$\begin{aligned} B &= (2b-d)d \\ B' &= (2a-c)c \\ D &= (2ad-cd-1) \\ D' &= (2bc-cd+1) \end{aligned}$$

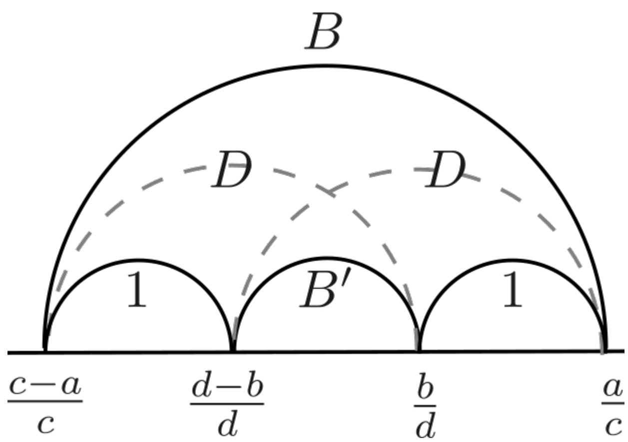

FIGURE 4. Proof that $1 + B \times B'$ is a square.

Note that $D - D' = 0$ since $ad - bc = 1$. Then

$$\begin{aligned}
B \times B' &= (2b-d)d \times (2a-c)c \\
&= 4abcd + (cd)^2 + 2bc^2d + 2acd^2 \\
&= 4abcd + (cd)^2 - 1 + 2bc^2d + 2acd^2 + 2 - 1 \\
&= 4abcd + (cd)^2 - 1 + 2bc^2d + 2acd^2 - cd + bc + 2(bc - ad) - 1 \\
&= (2ad - cd - 1)(2bc - cd + 1) - 1 \\
&= D \times D' - 1
\end{aligned}$$

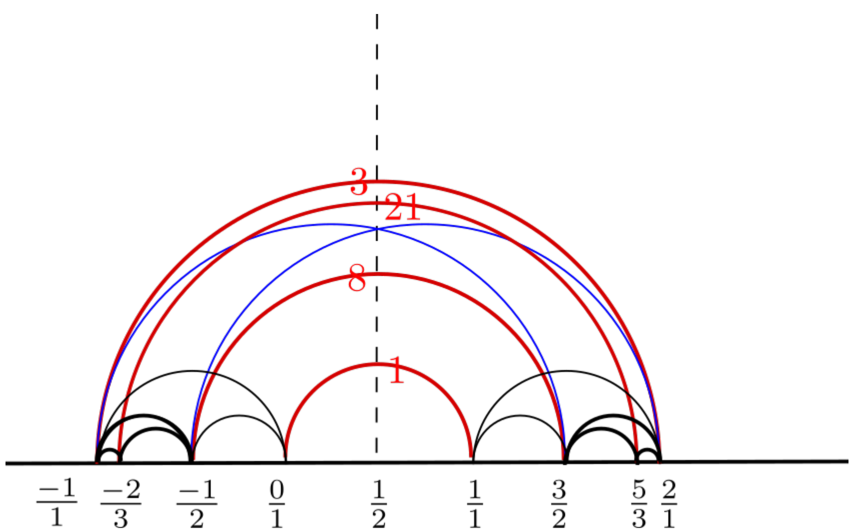

FIGURE 5. The next arc in the family.

3. FAMILIES OF TRIPLES

Fix some integer n that factorises as

$$n = b(2a - b) \tag{7}$$

with a, b a pair of coprime positive integers. By Bézout's lemma there are positive integers c, d such that

$$ad - bc = 1 \tag{8}$$

so that $\frac{a}{b}$ $\frac{c}{d}$ are vertices of an arc in the Farey diagram and this arc is an edge of an ideal triangle whose remaining vertex is the mediant of $\frac{a}{b}$ $\frac{c}{d}$. In fact one $\frac{a}{b}$ is the vertex of infinitely many ideal triangles with vertices consecutive elements of the sequence

$$\frac{c+ka}{d+kb}, k \in \mathbb{Z}.$$

By direct calculation one has:

Lemma 3.1. For any $k \in \mathbb{Z}$ the numbers

$$n, (d+kb)(2c-d+k(2a-b)), (b+d+kb)(2(a+c)-(b+d)+k(2a-b)),$$

form a Diophantine triple.

So for example if $n =$ then

$$b(2a-b) = 1,$$

has exactly one solution namely $a = b = 1$.

4. Parity of triples

There are two distinct families of Diophantine triples arising from the Farey diagram: those such that a, b, c are all multiples of 4 and those which are comprised of odd and even integers.

Lemma 4.1. The parity modulo 4 of the vertices of each ideal triangle is $0, 1, -1$

Proof. Each triangle is adjacent to exactly three others and so it suffices to suppose that the triangle has the stated parity and check that each of the neighbors has too. ☐

5. Quadruples

Perhaps the most famous Diophantine quadruple is $1, 3, 8, 120$ which is the unique extension of the triple $1, 3, 8$ to a quadruple (see [1]). As we have seen above the triple $1, 3, 8$ can be realised geometrically.

The triple $1, 3, 120$ cannot be realised geometrically but each of the tuples $1, 120$ and $3, 120$ can be realised geometrically.

There are exactly two ways (up to sign) of representing 120 by $a^2 - b^2$ namely $120 = 11^2 - 1^2$ and $120 = 13^2 - 7^2$. Each of these gives rise to a family of triples

$$n\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}11\\1\end{pmatrix}, m\begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}13\\7\end{pmatrix}$$

6. Appendix: realisable ideal triangles

Given a triple a, b, c of real numbers one can ask if there is a triangle with them as lengths. The answer is yes if and only if the triangle inequality holds that is:

$$a+b > c, b+c > a, c+a > b.$$

One can ask the same question for ideal triangles and λ-lengths: given a triple of real numbers $a, b, c \geq 1$ is there an ideal triangle with vertices in the extended rationals that has them as λ-lengths. The answer is yes if and only if the greatest common divisors of the three numbers are equal that is

$$\gcd(a, b, c) = \gcd(a, b) = \gcd(b, c) = \gcd(c, a).$$

This condition appears in the work of Cuntz, Holm and Jorgensen on Conway-Coxeter frieze patterns [3].

Lemma 6.1. Let $a, b, c \geq 1$ be a triple of integers then there is a rational ideal triangle with sides of λ-length a, b, c if and only if

$$\gcd(a, b, c) = \gcd(a, b) = \gcd(b, c) = \gcd(c, a).$$

Proof. Let a, b, c be a triple of numbers satisfying the hypotheses. Since the $\mathrm{PSL}(2, \mathbb{Z})$ action is transitive on the extended rationals we may assume that one of the vertices is $1/0$. Without loss of generality we suppose that the arcs asymptotic at $1/0$ have λ-lengths a and b respectively. This of course means that these arcs join $1/0$ to some pair of rationals y/a and x/b. The length of the arc joining this pair is

$$c = |ax - by|$$

Evidently c is a multiple of $\gcd(a, b)$ and by symmetry one obtains a is a multiple of $\gcd(b, c)$ and b is a multiple of $\gcd(a, c)$.

□

why doesn cuntz have a condition on the dyadic evaluation

7. Concluding remarks

We have given a geometric construction of Diophantine triples.One hopes that this will lead to a better understanding of Diophantine sets and their relation to the modular group.

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