FIBONACCI RELATIONS

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Abstract. Following Melhain and Shannon, Cheng and Mong [7] give a list of 12 identities involving Fibonacci numbers, which they attribute to various authors. We give a diagramatic interpretation (Figure 1) of these using λ-lengths of arcs on a hyperbolic annulus , and then we show how to derive them using essentially Penner's Ptolemy relation [8].

As always we claim no originality, and we are sure that this is well known to experts.

1. Introduction

The identities

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \tag{1}$$

and

$$F_{n-2}F_{n-1}F_{n+1}F_{n+2} - F_n^4 = -1 \tag{2}$$

According to Dickson [?], pages 393 and 401, the first of these identities was proved by Robert Simson and the second was stated by E. Gelin and proved by E. Cesàro. Simson's identity and the Gelin-Cesàro identity have been generalized many times.

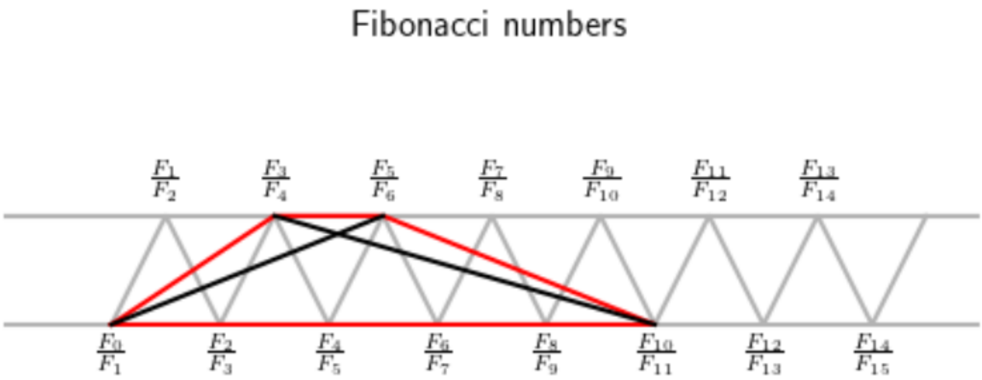

Figure 1. Fibonacci numbers

They prove the following variant of the Gelin-Cesàro identity:

(3) $$F_{n+1}F_{n+2}F_{n+6} - F_{n+3}^{n}F_n = (-1)^n F_n$$

1.1. **ChatGPT text conversion of Melhain and Shannon's introduction.** One method of proof is to use (1) to substitute for $(-1)^n$, then express each of F_{n-1}, F_n, F_{n+3}, and F_{n+6} in terms of F_{n+1} and F_{n+2}, and expand both sides. We prefer this method of proof since it carries over nicely to our generalization of (3), which we give next.

Our generalization is stated for the sequence $\{W_n\} = \{W_n(a, b; p, q)\}$ defined by

$$W_n = pW_{n-1} - qW_{n-2}, \quad W_0 = a, \quad W_1 = b$$

where a, b, p, q are taken to be arbitrary complex numbers with $q \neq 0$. Since $q \neq 0$, $\{W_n\}$ is defined for all integers n. Put $e = pab - qa^2 - b^2$. Then

(4) $$W_{n+1}W_{n+2}W_{n+6} - W_{n+3}^{3} = eq^{n+1}(p^3 W_{n+2} - q^2 W_{n+1})$$

2. Chen and Mong's list

Chen and Mong give a list of identities having done a search of the literature Long [3], and Weisstein [4] and they attribute these identities to various authors. Below is an abridged list omitting those entries which involve Lucas numbers and which are proven using induction.

(c1)
$$F_{n+a}F_{n+b} - F_nF_{n+a+b} = (-1)^n F_a F_b$$
(c4) $a = 1, b = -1$
$$F_{n-1}F_{n+1} - F_n^2 = (-1)^{n-1}$$
(d4)
$$F_n^2 - F_{n-2}F_{n+2} = (-1)^n$$
(d6)
$$F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1}$$
(c2)
$$F_{n+1}^2 = 4F_nF_{n-1} + F_{n-2}^2$$
(d1)
$$F_{2n} = F_{n+1}^2 - F_{n-1}^2$$
(c7)
$$F_{n+m} = F_{n-1}F_m + F_nF_{m+1}$$
(d2)
$$F_{2n+1} = F_{n+1}^2 + F_n^2$$
(d9)
$$\Rightarrow F_{2n+1} + (-1)^n = F_{n-1}F_{n+1} + F_{n+1}^2$$
Since $F_{n-1}F_{n+1} + (-1)^{-1} = F_n^2$.

(d12)
$$F_{2n+1} = F_{n+3}F_n - F_{n+1}F_{n-1}$$

(c9)
$$F_{2k+1}F_{2n+1} = F_{n+k+1}^2 + F_{n-k}^2$$

(d3)
$$F_{n+2}F_{n-1} = F_{n+1}^2 - F_n^2$$

(d7)
$$F_nF_{n+3} = F_{n+1}F_{n+2} + (-1)^{n-1}$$

(d8)
$$F_n^2 - F_{n-1}^2 = F_nF_{n-1} + (-1)^{n-1}$$

Is a bit cheeky too since: $F_n^2 - F_{n-1}(F_{n-1} + F_n) = F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1}$

(c11)
$$F_n^2 + (-1)^{n+r-1}F_r^2 = F_{n-r}F_{n+r}$$

(c12)
$$F_n^4 - F_{n-2}F_{n-1}F_{n+1}F_{n+2} = 1$$

3. Proofs

We'll give proofs grouping identities as per the previous section.

References

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