ISOSPECTRAL CONFIGURATIONS IN EUCLIDEAN AND HYPERBOLIC GEOMETRY

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1. Introduction

1.1. **Chord lengths.** The distribution of lengths of chords of convex bodies is a fundamental problem in integral geometry and has applications to tomography and x-ray crystallography. The basic question is: does the distribution of chord lengths determine the convex body up to isometry. By cleverly partitioning the sides of a regular octagon into two subsets X and Y Mallows and Clark [8] constructed a pair of convex dodecagons which are not congruent but have the same distribution of chord lengths. They obtain these dodecahedra by suitably capping off each of the sides of X (resp. Y) with a triangle.

What is important in the Mallows Clark construction, as was observed by R. Garcia-Pelayo [7], is that the configuration of sides in X and Y are associated to *non congruent homometric pairs* (NCHP) in the cyclic graph C_8. He further observes that, after applying an isometry of $\mathbb{R}^2$ the sets X and Y can be chosen to be complementary. Garcia-Pelayo goes on to show that any pair of complementary sets, of a finite vertex transitive graph are in fact homometric thus generalizing a theorem of A. L. Patterson from the 1940s. Here by *pair of complementary sets* we mean a pair of subsets of vertices of the same cardinal whose reunion is the set of all vertices.

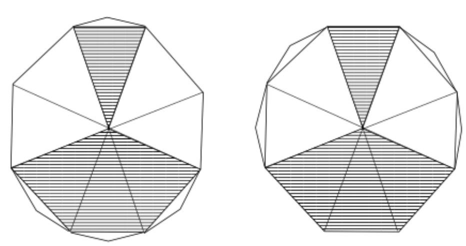

Figure 1. Mallows Clark pair.

1.2. **Spanning rulers.** In fact, NCHP have been extensively studied in relation to spanning rulers. A *ruler* is any finite subset of the positive integers containing 0 and the points of the set are called *marks.* A *spanning ruler* is a ruler such that the distances between (distinct) points are distinct. More informally a *regular ruler* consists of the points$\{0, 1, 2 \ldots n\}$ so that the distance 1 can be measured in $n-1$ ways (e.g $2-1$, $3-2$ etc.) whilst on a spanning ruler each distance that can be

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measured can be done so in exactly one way. An example of a spanning ruler is $0, 1, 4, 6$ which measures all the distances $1, 2, 3, 4, 5, 6$ just once. There are diverse applications of spanning rulers ranging from in radio astronomy to x-ray diffraction crystallography.

In 1939 Sophie Piccard claimed that for any spanning ruler the set of distances it measures determines it up to isometry or that, equivalently, any ruler from an NCHP is never a spanning ruler. In the 70's G. Bloom [3] found an NHCP which is a counterexample to her claim, namely the pair $\{0, 1, 4, 10, 12, 17\}$ and $\{0, 1, 8, 11, 13, 17\}$. Subsequently, Bloom-Golomb [4] found an infinite family of counterexamples each with six marks containing this pair.

1.3. **Structure and results.** Our principal motivation was an attempt to find pairs of hyperbolic surfaces that had the same chord distribution which are not isometric and perhaps not even commensurable (i.e. they have no common cover of finite degree. Thus, with the exception of Sections 5 and 6 which have some geometric/topological content most of this text is expository.

We begin (Section 2 and 3) by reformulating Garcia-Pelayo's result in terms of auto-correlation functions which will allow us to present a short proof of them. In Section 4 we discuss a particular family of NHCP using this approach. Further we construct NCHP which are counterexamples to the questions asked by Garcia-Pelayo. In particular we find NCHP pairs for n-polygons for $n > 15$ odd using the topological ideas introduced in Section 6.

In Section 5 we adapt the construction of Mallows Clark in the hyperbolic setting yielding examples of non congruent ideal polygons with the same distribution of lengths of chords. We also construct other pairs of hyperbolic surfaces with the same distribution of lengths of chords. All our examples are of rather simple topological type being homeomorphic to annuli. It seems difficult to construct non isometric examples of pairs of more general surfaces with the same distribution of chord lengths; see for example the discussion in [9]. Nonetheless even such simple surfaces have appeared in several papers in the context of hyperbolic geometry and identities for the Roger's Dilogarithm [10, 5, 11].

We end the paper with two appendices with which we hope to tempt the reader to engage further in the problem of construction and classification of NHCP. The first describes the Mallows-Clark construction for NHCP pairs in geometric terms and the second Golomb's algebraic approach using what amounts essentially to factorisation of the auto-correlation function.

2. Generating and auto-correlation functions

Let G be a connected graph with vertices $V(G)$. There is a natural *path metric* on G with distance function $d : V(G) \times V(G) \to \mathbb{R}_+$ such that for adjacent vertices $v_1, v_2 \in V(G)$ the distance satisfies $d(v_1, v_2) = 1$. It is easy to see that any automorphism of G is an isometry of this metric. To each vertex $v \in V(G)$ we associate a *generating series* namely

$$\sigma_v(t) = \sum_{u \in V(G)} t^{d(v,u)}.$$

If $(X_i)_i$ is a decomposition of $V(G)$ that is

$$V(G) = \bigsqcup_i X_i,$$

then σ_v decomposes as

$$\sigma_v(t) = \sum_i \left(\sum_{u \in X_i} t^{d(v,u)} \right).$$

More generally define the *auto-correlation function* of $X \subset V(G)$ to be

$$\sigma_X(t) = \frac{1}{2} \sum_{v \in X} \sigma_v(t) = \frac{1}{2} \sum_{(v,u) \in X^2} t^{d(v,u)}.$$

Let G be a connected graph and T a fixed point free automorphism of G then there is an obvious relation between these series: If $X \subset V(G)$ is such that $V(G) = X \sqcup T(X)$ then by letting $\overline{X} := V(G) \setminus X = T(X)$, we have

$$\sigma_X = \sigma_{\overline{X}}.$$

3. Swapping in vertex transitive graphs

Recall that a graph G is *vertex transitive* iff its automorphism group acts transitively on the vertices $V(G)$. Note that a vertex transitive graph must be *regular*, that is every vertex has the same valence. Since any automorphism is an isometry of the natural path metric one has, for all $u, v \in V(G)$,

$$\sigma_u(t) = \sigma_v(t).$$

Let X be a subset of $V(G)$ and $\overline{X}$ denote the complement of X. A set Y is obtained from X by *swapping* $u \in X$ for $v \in \overline{X}$ iff

$$X = Z \cup \{u\},\ Y = Z \cup \{v\},$$

where $Z = X \cap Y$. Note that $\overline{Y}$ is obtained from $\overline{X}$ by swapping v for u since

$$\overline{X} = Z' \cup \{v\},\ \overline{Y} = Z' \cup \{u\},$$

where $Z' = \overline{X} \cap \overline{Y}$. If X_0 and X_1 are a pair of finite subsets with the same number of elements then one can transform X_0 into X_1 by a finite number of swaps. The minimal number of swaps is called the *Hamming distance*.

Theorem 3.1. *Let G be a connected finite vertex transitive graph and $X \subset V(G)$ and $\overline{X}$ denote the complement of X. Then for any subset Y obtained from X by a finite number of swaps*

$$\sigma_X - \sigma_{\overline{X}} = \sigma_Y - \sigma_{\overline{Y}}.$$

Proof. The proof is by induction on the number of *swaps.*

It is easy to see that it suffices to prove the theorem for a single swap. We present a proof which is completely formal as follows. There is an obvious decomposition of the spectral functions:

$$\sigma_X = \sigma_Z + \sum_{x \in Z} t^{d(u,x)} + \frac{1}{2}$$

$$\sigma_{\overline{X}} = \sigma_{Z'} + \sum_{x \in \overline{Z'}} t^{d(v,x)} + \frac{1}{2}$$

likewise

$$\sigma_Y = \sigma_Z + \sum_{x\in Z} t^{d(v,x)} + \frac{1}{2}$$
$$\sigma_{\overline{Y}} = \sigma_{Z'} + \sum_{x\in \overline{Z'}} t^{d(u,x)} + \frac{1}{2}$$

Thus

$$\sigma_X - \sigma_{\overline{X}} = \sigma_Z - \sigma_{Z'} + \left(\sum_{x\in Z} t^{d(u,x)} - \sum_{x\in Z'} t^{d(v,x)}\right) \tag{1}$$

$$\sigma_Y - \sigma_{\overline{Y}} = \sigma_Z - \sigma_{Z'} + \left(\sum_{x\in Z} t^{d(v,x)} - \sum_{x\in Z'} t^{d(u,x)}\right). \tag{2}$$

The generating functions also decompose as

$$\sigma_u(t) = \sum_{x\in Z} t^{d(u,x)} + \sum_{x\in Z'} t^{d(u,x)} + t^{d(u,v)}$$
$$\sigma_v(t) = \sum_{x\in Z} t^{d(v,x)} + \sum_{x\in Z'} t^{d(v,x)} + t^{d(u,v)}$$

By hypothesis $\sigma_u(t) = \sigma_v(t)$ so subtracting these expressions one sees that the expressions in parentheses in equations (1) and (2) are equal. □

Corollary 3.2. *Let G be a connected finite vertex transitive graph. If $\sigma_X = \sigma_Y$ then $\sigma_{\overline{X}} = \sigma_{\overline{Y}}$.*

Proof. By the theorem

$$\begin{aligned}\sigma_X - \sigma_{\overline{X}} &= \sigma_Y - \sigma_{\overline{Y}} \\ &= \sigma_X - \sigma_{\overline{Y}}\end{aligned}$$

so the result follows. □

Corollary 3.3. *Let G be a connected finite vertex transitive graph with $2n$ vertices. Suppose G has a fixed point free automorphism T of even order. Then for any subset of X of n vertices*

$$\sigma_X = \sigma_{\overline{X}}.$$

Proof. Since T is of even order there exists a subset $X_0 \subset V(G)$, such that $T(X_0) = \overline{X_0}$. Further T being a bijection, the sets $T(X_0)$ and X_0 have the same number of elements, so in particular X_0 must have exactly n elements. Since T is an isometry of the path metric $\sigma_{X_0} = \sigma_{\overline{X}_0}$.

Since $V(G)$ is finite one can transform any other subset X of n element into X_0 by a finite number of swaps so it follows from Theorem 3.1 that

$$0 = \sigma_{X_0} - \sigma_{\overline{X}_0} = \sigma_X - \sigma_{\overline{X}}.$$

□

4. Pairs of subset

Using Corollary 3.3 it is easy to find pairs of sets X and Y such that $\sigma_X = \sigma_Y$ but which are not congruent. Here we say that X and Y are *congruent* if there is an automorphism of G mapping X onto Y. In fact, it is also easy to find pairs of sets X and Y such that $\sigma_X = \sigma_Y$ and Y is congruent to neither X nor $\overline{X}$.

Let G be a cycle of $2n$ vertices and X a subset of n vertices such that $\overline{X}$ is not congruent to X. Let $2G$ denote the graph obtained from G by subdividing each edge, it is a cycle of $4n$ vertices. There is a natural inclusion $\iota : V(G) \hookrightarrow V(2G)$ and, since $d(\iota(u), \iota(v)) = 2d(u, v)$, one has

$$\sigma_{\iota(X)}(t) = \sigma_X(t^2), \;\; \sigma_{\iota(\overline{X})}(t) = \sigma_{\overline{X}}(t^2).$$

But $\iota(\overline{X})$ has n vertices and so is not congruent to the complement of $\iota(X)$ which has $3n$.

4.1. **Homometric configurations on the line.** For an n-point configuration $X = \{x_0, \ldots, x_{n-1}\}$ the distance spectrum $\mathcal{D}(X)$ is defined as the set

$$\{|x_i - x_j| \mid 0 \leq i < j \leq n-1\},$$

counted with multiplicities. It is convenient to label the points so that $x_i < x_{i+1}$ for all $0 \leq i \leq n-1$. Two configurations are equivalent if they are related by a finite sequence of translations, dilations, and reflections of the line. The following proposition provides us with a recursively defined infinite family of inequivalent homometric configurations.

Proposition 4.1. *For every $k \geq 12$, the sets $X_k := \{0, 1, 5, 6, 7, 9, 10..., k-4, k-3, k\}$ and $Y_k := \{0, 4, 5, 6, 9, 10, ..., k-4, k-3, k-1, k\}$ are $(k-5)$-point inequivalent homometric configurations.*

Proof. Note that

$$X_{k+1} = X_k \setminus \{k\} \cup \{k-2, k+1\}$$
$$Y_{k+1} = Y_k \setminus \{k-1\} \cup \{k-2, k+1\}.$$

Hence to compute $\mathcal{D}(X_{k+1}) \setminus \mathcal{D}(X_k)$ and $\mathcal{D}(Y_{k+1}) \setminus \mathcal{D}(Y_k)$, it suffices to deal with three points in each case. Thus it can be seen that both $\mathcal{D}(X_{k+1}) \setminus \mathcal{D}(X_k)$ and $\mathcal{D}(Y_{k+1}) \setminus \mathcal{D}(Y_k)$ consist of the points

$$\{1, 2, \ldots, k-11, k-9, k-8, k-8, k-4, k-3, k-2, k+1\} - \{k-1\}.$$

□

4.2. **Configurations of five or less points.** It is known that there are no homometric pairs with less than six points. We give a short proof of this fact for completeness. Let $X = \{v_1, \ldots, v_5\}$ be five points on the line arranged in order according to the index and note that there are exactly 10 elements in the set of distances $\mathcal{D}(X)$. Note that $|v_5 - v_1|$ is the maximum of $\mathcal{D}(X)$ and that, with the exception of $|v_5 - v_1|$ and $|v_4 - v_2|$, the distances can be paired off (see Figure 2) so that the sum of a pair is either $|v_5 - v_1|$ or $|v_4 - v_2|$ thus:

$$2|v_4 - v_2| + 4|v_5 - v_1|. = \sum_{l \in \mathcal{D}(X)} l \tag{3}$$

The second largest distance in $\mathcal{D}(X)$ is either $v_5 - v_2$ or $v_4 - v_1$. and since we know the value of $v_4 - v_2$ from formula (3) above we can determine v_2 and v_4 and the

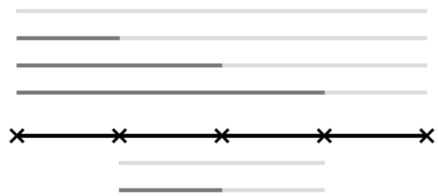

FIGURE 2. 5 points configuration.

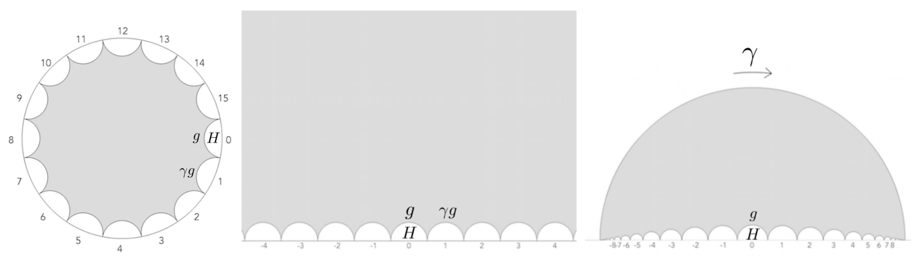

FIGURE 3. (left) elliptic, (center) parabolic, (right) hyperbolic.

the 6 distances $|v_i - v_j|$, $i > j, i \neq 4, j \neq 3$. After removing these distances there are 2 pairs of distances involving v_3 and we can determine v_3 from the equations:

$$\begin{aligned} |v_3 - v_2| + |v_3 - v_4| &= |v_4 - v_2| \\ |v_3 - v_1| + |v_3 - v_5| &= |v_5 - v_1|. \end{aligned}$$

5. ORTHO SPECTRUM

We consider the hyperbolic plane $\mathbb{H}^2$. Let $\mathcal{G}$ be a collection of mutually disjoint (possibly asymptotic) geodesics. Two such collections are *congruent* if there is an isometry which maps one to the other set-wise. An *ortho geodesic* is a geodesic each of two endpoints is orthogonal to an element of $\mathcal{G}$. The *ortho spectrum* $\mathcal{O}(\mathcal{G})$ is the set of the lengths of ortho geodesics counted with multiplicity. For a hyperbolic surface with totally geodesic boundary, we get a collection of geodesics as the boundary of the universal covering, and ortho spectrum is defined to be the one for those geodesics. In this section, by using Corollary 3.3, we construct examples of incongruent collections of geodesics with the same ortho spectrum. Some of those examples also give examples of non-isometric hyperbolic surfaces with the same ortho spectrum. In [9], we further discuss systoles and ortho spectrum rigidity. First, let g be a geodesic, and γ be an isometry which satisfies the following condition.

Condition 5.1. For all $i \in \mathbb{Z}$,

- we have either $g \cap \gamma^i g = \emptyset$ or $g = \gamma^i g$, and
- $\gamma^i g$'s are in the same component of the complement of g.

Let H be the half space with respect to g that does not contain $\gamma^i g$'s. Let $\mathcal{G}_0$ be a collection of disjoint geodesics in H. Then put $\mathcal{G}_i := \gamma^i(\mathcal{G}_0)$ for $i \in \mathbb{Z}$.

For two disjoint collections of geodesics $\mathcal{H}_1$ and $\mathcal{H}_2$, let $\mathcal{O}(\mathcal{H}_1, \mathcal{H}_2)$ be the lengths of all ortho geodesics connecting $\mathcal{H}_1$ and $\mathcal{H}_2$, counted with multiplicity. The following proposition is immediate from the construction.

Proposition 5.2. *$\mathcal{O}(\mathcal{G}_{i_1}, \mathcal{G}_{i_2}) = \mathcal{O}(\mathcal{G}_{j_1}, \mathcal{G}_{j_2})$ whenever $|i_1 - i_2| = |j_1 - j_2|$.*

Let C_{2n} be the cycle graph of length $2n$ and $\widetilde{C_{2n}}$ a universal covering of C_{2n}. We choose a generator t of covering transformation and label $V(\widetilde{C_{2n}})$ so that for each $i \in \mathbb{Z}$,

- $t(v_i) = v_{i+2n}$, and
- v_{i-1} and v_i is connected by an edge.

Let X' be an n-point subset of $V(C_{2n})$ and X a lift to $\widetilde{C_{2n}}$ of X'.

We define two collections of geodesics by

$$\mathcal{G}_X := \bigcup_{v_i \in X} \mathcal{G}_i,$$
$$\mathcal{G}_{\overline{X}} := \bigcup_{v_i \in \overline{X}} \mathcal{G}_i.$$

Then we have

Theorem 5.3. *Two collections $\mathcal{G}_X$ and $\mathcal{G}_{\overline{X}}$ have the same ortho spectrum, that is $\mathcal{O}(\mathcal{G}_X) = \mathcal{O}(\mathcal{G}_{\overline{X}})$.*

Proof. Let $I_X := \{(i,j) \mid i < j, v_i, v_j \in X\}$. If $\gamma^m g = g$ for some m, then we only consider $0 \le i < j < m$, see also Corollary 5.5 below. The ortho spectrum can be decomposed as

$$\mathcal{O}(\mathcal{G}_X) = \bigcup_{v_i \in X} \mathcal{O}(\mathcal{G}_i) \cup \bigcup_{(i,j) \in I_X} \mathcal{O}(\mathcal{G}_i, \mathcal{G}_j).$$

Hence by Proposition 5.2, we see that $\mathcal{O}(\mathcal{G}_X)$ is determined by the spectral series of X. Note that Proposition 5.2 includes the case where $|i - j| = 0$. By Corollary 3.3, X and $\overline{X}$ have the same spectral series and hence we have $\mathcal{O}(\mathcal{G}_X) = \mathcal{O}(\mathcal{G}_{\overline{X}})$. □

Corollary 5.4. *There are incongruent collections of geodesics with the same ortho spectrum.*

Proof. Any incongruent pair X' and $\overline{X'}$ of n-point subsets of $V(C_{2n})$ gives a incongruent pair of geodesics $\mathcal{G}_X$ and $\mathcal{G}_{\overline{X}}$ with the same ortho spectra. Incongruent pairs of n-point subsets of $V(C_{2n})$ are given in [8, 7]. □

According to the type of the isometry γ, we have three different kinds of examples. Let H_X (resp. $H_{\overline{X}}$) be the connected component of $\mathbb{H}^2 \setminus \mathcal{G}_X$ (resp. $\mathbb{H}^2 \setminus \mathcal{G}_{\overline{X}}$) whose boundary contains whole $\mathcal{G}_X$ (resp. $\mathcal{G}_{\overline{X}}$).

Corollary 5.5. *There are non-isometric hyperbolic orbifolds with the same ortho spectrum.*

Proof. We suppose that γ is elliptic. Then Condition 5.1 implies that γ is of finite order. Let m be the smallest integer so that $\gamma^m g = g$. We also consider the case where γ is identity, or $m = 0$. We assume that $2n$ divides m. Both H_X and $H_{\overline{X}}$ are symmetric with respect to γ^{2n}. Hence $H_X/\langle\gamma^{2n}\rangle$ and $H_{\overline{X}}/\langle\gamma^{2n}\rangle$ are hyperbolic orbifolds. It is easy to see that if X and $\overline{X}$ are incongruent, then $H_X/\langle\gamma^{2n}\rangle$ and $H_{\overline{X}}/\langle\gamma^{2n}\rangle$ are non-isometric.

Corollary 5.6. *There are non-isometric hyperbolic surfaces with the same ortho spectrum.*

Proof. We give two kinds of examples. First, suppose γ is parabolic. We assume that the endpoints at infinity of g are not parabolic fixed point of γ. Then for any m with $2n|m$, $H_X/\langle\gamma^m\rangle$ and $H_{\overline{X}}/\langle\gamma^m\rangle$ are punctured surfaces with the same ortho spectrum. They are non-isometric whenever X and $\overline{X}$ are incongruent.

If γ is hyperbolic, we assume that the geodesic axis h does not intersect with g. Then let H' be the half space with respect to h which contains g. Then for any m with $2n|m$, $(H_X \cap H')/\langle\gamma^m\rangle$ and $(H_{\overline{X}} \cap H')/\langle\gamma^m\rangle$ are hyperbolic surfaces with one totally geodesic boundary. They have the same ortho spectrum, and again they are non-isometric whenever X and $\overline{X}$ are incongruent. □

6. Projecting from $\mathbb{Z}$ onto C_n

In this section we show how to get examples of non congruent homometric pairs of subsets of the cyclic graph on n vertices using covering maps. In particular we obtain subsets which are not pairs of complementary sets as in Section **??**. Further we find NCHP pairs for n-polygons for $n > 15$ odd using the topological ideas introduced in the previous section.

6.1. **Distances and covering maps.** Let C_n be the cyclic graph on n vertices and C_∞ the connected infinite graph all of whose vertices are valence 2. We can identify C_∞ with the universal cover of C_n and there is a projection map $\pi : C_\infty \to C_n$. One can view these graphs as Cayley graph of (cyclic) groups in an obvious way: C_∞ is the Cayley graph of $(\mathbb{Z}, +)$ with respect to the generator 1 and C_n of $(\mathbb{Z}/n\mathbb{Z}, +)$. In this way $\mathbb{Z}$ is naturally identified with the vertices of C_∞ and $\mathbb{Z}/n\mathbb{Z}$ with those of C_n. In particular we can identify the vertices of the latter with $0, 1, \dots n-1$ in the usual way.

Let $x_i, y_i \in C_\infty$, if $d(x_1, y_1) = d(x_2, y_2)$ then $d(\pi(x_1), \pi(y_1)) = d(\pi(x_2), \pi(y_2))$. Since C_∞ is a vertex transitive it suffices to fix x and to check that there is a function F such that for every vertex y of C_∞

$$d(\pi(x), \pi(y)) = F(d(x, y)).$$

One checks that F is the absolute value of the symmetric remainder for the division by n. Recall that *symmetric remainder* is the integer r such that $p = qn + r$ and $|r| < n/2$.

6.2. **Images of homometric pairs under covering maps.** An n point configuration X is a map $\iota_X : \{1, 2, \dots n\} \to \mathbb{Z}$. It is convenient to work with configurations as they correspond to subsets of points counted with multiplicities. We say that a point x of X is a *multi point* iff $\{x\}$ is strictly contained in $\iota_X^{-1}(\{\iota_X(x)\})$. Two configurations ι_X and ι_Y are isometric iff there exists g an isometry of $\mathbb{Z}$ such that $g \circ \iota_X = \iota_Y$. Any subset of $\mathbb{Z}$ can be viewed as a configuration with no multi points.

If $\tilde{X}, \tilde{Y}$ is a pair of non congruent homometric subsets of $\mathbb{Z}$ then, by the above discussion, under the projection map π the resulting configurations $\pi(\tilde{X})$ and $\pi(\tilde{Y})$ are homometric. The examples we obtain by this method fall into three classes:

(1) If n is greater than twice the diameter of $\tilde{X}$ then the restriction of the projection π to $\tilde{X}$ (resp. $\tilde{Y}$) is injective and, moreover, this restriction maps $\tilde{X}$ isometrically onto $\pi(\tilde{X})$. Thus in this case we obtain a pair of non congruent homometric subsets of C_n;

(2) If n is greater than the diameter of $\tilde{X}$ but less than twice the diameter of $\tilde{X}$ then the restriction of the projection π to $\tilde{X}$ (resp. $\tilde{Y}$) is injective but $\tilde{X}$ is not mapped isometrically onto $\pi(\tilde{X})$. Thus we obtain a pair of (possibly congruent) homometric sets of C_n.

(3) If n is less than the diameter of $\tilde{X}$ then the restriction of π is not a priori injective so we obtain (possibly congruent) homometric configurations.

6.3. **An example.** The sets $\tilde{X} = \{0, 2, 5, 14, 18, 25\}$ and $\tilde{Y} = \{0, 2, 13, 16, 20, 25\}$ are non congruent homometric subsets of $\mathbb{Z}$ and under the projection to $\mathbb{Z}/n\mathbb{Z}$ for $n = 11, 12, 13, 14$ they give configurations each with a single double point. For $n = 15$ one obtains a pair of sets $X = \{0, 2, 3, 5, 10, 14\}, Y = \{0, 1, 2, 5, 10, 13\}$. These sets are not congruent in C_{15} since the distances between consecutive points are different: these are respectively $2, 1, 2, 3, 4, 1$ and $1, 1, 3, 5, 3, 2$.

Thus we obtain a table of pairs of NCHP:

15	0, 2, 3, 5, 10, 14	0, 1, 2, 5, 10, 13
17	0, 1, 2, 5, 8, 14	0, 2, 3, 8, 13, 16
19	0, 2, 5, 6, 14, 18	0, 1, 2, 6, 13, 16
21	0, 2, 4, 5, 14, 18	0, 2, 4, 13, 16, 20
22	0, 2, 3, 5, 14, 18	0, 2, 3, 13, 16, 20
24	0, 1, 2, 5, 14, 18	0, 1, 2, 13, 16, 20

For C_{21} the two sets we obtain are $0, 2, 4, 5, 14, 18$ and $0, 2, 4, 13, 16, 20$ for which the distances between consecutive points are respectively 2,2,1,9,4,3 and 2,2,9,3,4,1. So these sets are not isometric even though (setwise) these distances between consecutive points are the same.

7. Remarks, further questions

As indicated in the introduction, in Section 5 we succeeded in constructing pairs of hyperbolic surfaces with the same distribution of lengths of chords. Our examples are of rather simple topological type being homeomorphic to annuli and in particular what Casson and Bleier [6] refer to as crowns - that is there is a boundary component that consists of a single closed geodesic and the other boundary is a union of complete geodesics asymptotic in a finite numberof spikes.

Question: Is it possible to construct a pair of non isometric surfaces, each of which consists of a pair of crowns identified along the boundary components that are closed geodesics, but which have a common orthospectrum?

8. Appendix: Structure of homometric pairs

In this section we study the underlying structure of the Mallows-Clark pair decomposing it into three sets X, Y and $P_{\pm}$. Our decomposition will allow us to show that:

- the Mallows-Clark pair lies in a one parameter family of NCHP of the circle.
- to construct, for each $n \geq 4$, NCHP in the n-sphere $\mathbb{S}^n$.

8.1. **Structure of the Mallows-Clark pair.** Each of the two configurations C_1, C_2 of the Mallows-Clark pair consists of four points and for each of the configurations exactly two of these points lie on a diameter of the circle We can apply a rotation so that $C_1 \cap C_2$ consists of exactly three points: a pair on the diameter x_1, x_2 and another point y. The set $C_1 \setminus C_2$ consists of a single point p_1 likewise $C_2 \setminus C_1$ is just p_2. One verifies that p_1, p_2 are also on a diameter. Thus

$$\begin{aligned} C_1 \cup C_2 &= X \sqcup Y \sqcup P \\ C_1 &= X \sqcup Y \sqcup \{p_1\} \\ C_2 &= X \sqcup Y \sqcup \{p_2\} \end{aligned}$$

where $X = \{x_1, x_2\}$, $Y = \{y\}$ and $P = \{p_1, p_2\}$.

Let Ant denote the antipodal map of the circle and Refl the unique orientation reversing map that has y as its fixed point. Both of these maps are isometries of the circle. Then:

(1) X is invariant under the antipodal map Ant;
(2) Y is invariant under Refl;
(3) P is invariant under both Ant and Refl and each of these maps exchange p_1 and p_2 that is

$$\mathrm{Ant}(p_1) = p_2,\ \mathrm{Refl}(p_1) = p_2. \tag{4}$$

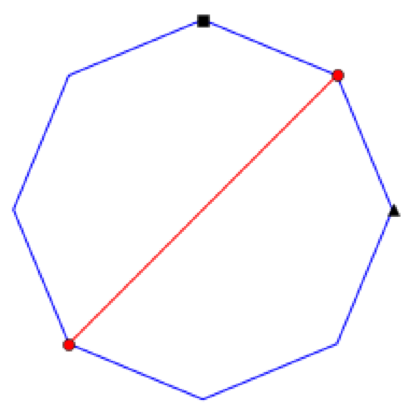

FIGURE 4. One of the configurations of a Mallows-Clark pair. The black dots are antipodeal pairs. The black triangles is a point invariant under the reflection Refl. The square is one point of an antipodeal pair invariant under reflection in the diameter.

It is easy to verify that the configurations are homometric without doing any calculations as follows. Since $C_1 \cap C_2 = X \sqcup Y$ it suffices to show that the pairs $X \sqcup \{p_1\}$, $X \sqcup \{p_2\}$ and $Y \sqcup \{p_1\}$, $Y \sqcup \{p_2\}$ are isometric (and so homometric). We have

$$\begin{aligned} \mathrm{Ant}(X \sqcup \{p_1\}) &= X \sqcup \{\mathrm{Ant}(p_1)\} \\ \mathrm{Refl}(Y \sqcup \{p_1\}) &= Y \sqcup \{\mathrm{Refl}(p_1)\} \end{aligned}$$

8.2. **Deforming the Mallows-Clark pair.** Using the decomposition it is easy to construct a one parameter family of NCHP as follows.

The antipodeal map commutes with every isometry of the circle and in particular with the one parameter group of rotations R_t. In particular, if X is invariant under the antipodeal map then so is $R_t(X)$. For $t \in \mathbb{R}$ define

$$\begin{aligned} C_1^t &:= R_t(X) \sqcup Y \sqcup \{p_1\} \\ C_2^t &:= R_t(X) \sqcup Y \sqcup \{p_2\}. \end{aligned}$$

Thus we have a family such that, C_1^0 and C_2^0 is the original Mallows-Clark pair and for all $t \in \mathbb{R}$ the configurations C_1^t and C_2^t are homometric and non conjugate if $R_t(X)$ is not invariant under Refl.

8.3. **Construction of homometric pairs via pairs of involutions.** We can now give a quite general construction for homometric pairs of subsets C_1, C_2 of a metric space$A$. It is interesting to note that this construction can be applied to Euclidean space to obtain pairs such that the Minkowski difference is not the same, that is

$$C_1 - C_1 \neq C_2 - C_2.$$

where the difference is defined by

$$X - X := \{x - y,\, x, y \in X\}.$$

Theorem 8.1. *Let A be a metric space and α_1, α_2 a pair of involutions of A acting by isometry.*

If X_1, X_2, P_1, P_2 are disjoint subsets of A such that for $i = 1, 2$

$$\alpha_i(X_i) = X_i \tag{5}$$

$$\alpha_i(P_1) = P_2 \tag{6}$$

then the sets

$$C_i := (X_1 \sqcup X_2) \sqcup P_i,\, i = 1, 2$$

are homometric.

Further if

1. *α_1 is the restriction of a unique isometry of A.*
2. *X_1 is the maximal subset of C_1 invariant under α_1*
3. *X_1 is the unique subset of $C_1 \cup C_2$ conjugate to X_1*

then C_1 and C_2 are not conjugate.

The conditions in the second part of the theorem are sufficient but not necessary. They hold for original Mallows Clark pair where α_1 is the antipodeal map and X_1 the antipodeal pair, that is the unique set of points for which the diameter of C_i is attained. So X_1 is invariant under any conjugation of C_1 and C_2.

In Paragraph 8.5 we will construct a pair which while not conjugate does not satisfy these conditions. That is, there is X_1 conjugate to different subsets of $X_1 \cup P_1$. We will exploit this to produce a triple of homometric pairwise non conjugate configurations.

Proof. The proof of homometry is almost exactly as in Paragraph 8.1. Since $C_1 \cap C_2 = X_1 \sqcup X_2$ it suffices to show that, for $i = 1, 2$, the pairs $X_i \sqcup P_1$, $X_i \sqcup P_2$ are isometric (and so homometric). Consider, as before,

$$\alpha_i(X_i \sqcup P_1) = X_i \sqcup \alpha_i(P_1)$$

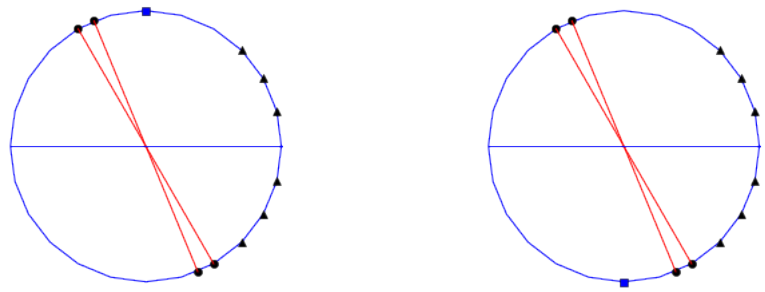

FIGURE 5. The configurations of a generalized Mallows-Clark pair. The black dots are antipodeal pairs. The black triangles points of a set invariant under reflection in the horizontal diameter. The squares are a pair of antipodeal points invariant reflection in the diameter.

for $i = 1, 2$, and by hypothesis α_i exchanges P_1 and P_2 so that $\alpha_i(P_1) = P_2$.

Now suppose that C_1 and C_2 are conjugate by an (necessarily non trivial) isometry α. This isometry conjugates X_1 to a subset of C_2 so, by hypothesis, $\alpha(X_1) = X_1$ and, since α_1 is the restriction of a unique isometry,

$$\alpha(X_2 \cup P_1) = \alpha_1(X_2 \cup P_1) = \alpha_1(X_2) \cup P_2$$

so that $\alpha_1(X_2) = X_2$ contradicting the maximality of X_1

□

8.4. **Two constructions of (generalized) Mallows-Clark pairs.** We now use Theorem 8.1 to construct the Mallows-Clark pair in two different ways. Let A be the circle $\{|z| = 1, z \in \mathbb{C}\}$ with the induced metric. Now we make two different choices for the involutions

(1) using the antipodeal map and a reflection $\alpha_1 : z \mapsto -z$, $\alpha_2 : z \mapsto \bar{z}$;
(2) using a pair of distinct reflections $\alpha_1 : z \mapsto -\bar{z}$, $\alpha_2 : z \mapsto \bar{z}$.

Note that all three of these involutions are restrictions of involutions of the complex plane. For the original Mallows-Clark pair this yields two different families of NCHP whose intersection consists of those configurations such that $P_1 \cup P_2$ lie on the pair of lines $y = \pm x$.

Corollary 8.2. *For either of the choices of involution above the pair of configurations arising from the construction in Theorem 8.1 lies in a real analytic family of homometric pairs. For almost all values of the parameters the resulting pairs are non conjugate.*

Proof. (Sketch) Existence follows from the theorem. The almost all values part is a consequence of the fact that the family is real analytic. □

Since the antipodeal map has no fixed points one sees that, for any family obtained from the construction of Corollary 8.2, X_1 must have an even number of elements whilst there is no restriction on the parity of either X_2 or the P_i. With this observation it is easy to see that:

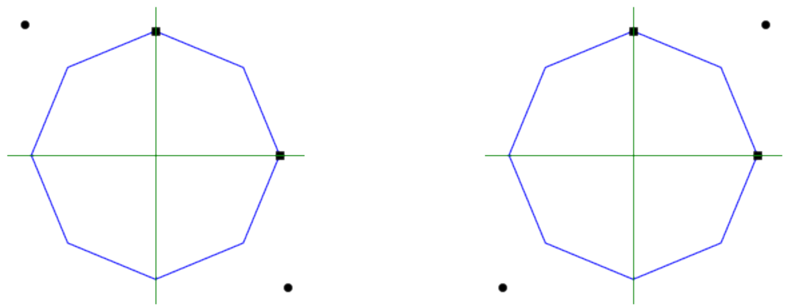

FIGURE 6. A (generalized) Mallows-Clark pair in the complex plane.

Corollary 8.3. *For any $n \geq 4$ there is a family of homometric pairs on the circle and for almost all values of the parameters the resulting pairs are non conjugate.*

8.5. **NCH Triples.** In this paragraph we construct a triple of homometric pairwise non conjugate configurations. Each pair of configurations does not satisfy the conditions of Theorem 8.1: the set X_1 will be conjugate to different subset X_1' of $X_1 \cup P_2$. The subset X_1' is invariant under a reflexion $\alpha_1' \neq \alpha_1$ and one defines P_1' to be $(X_1 \cup P_2) \setminus X_1'$. Then using the construction of Theorem 8.1 one obtains a set

$$C_3 = X_1' \sqcup X_2 \sqcup \alpha_1'(P_1').$$

which is homometric to $C_2 = X_1 \sqcup X_2 \sqcup P_2$. This procedure is illustrated in Figure 7 and Figure 8. Since the conditions of the second part of Theorem 8.1 no longer hold one must verify that the three configurations are pairwise non conjugate. First, observe that each of the pairwise intersections of the configurations is a unique antipodeal pair X_2 so that, if any pair is conjugate, the conjugation must be the antipodeal map α_2. Now it is easy to see that for the antipodeal map $\alpha_2(C_i) \neq C_j$.

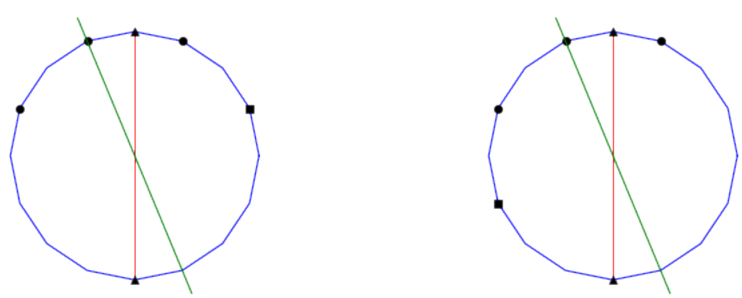

FIGURE 7. A generalized Mallows-Clark pair associated a reflexion α_1 and the antipodeal map α_2. The invariant set of α_1 consists of three black dots and p_1 is represented by a black square.

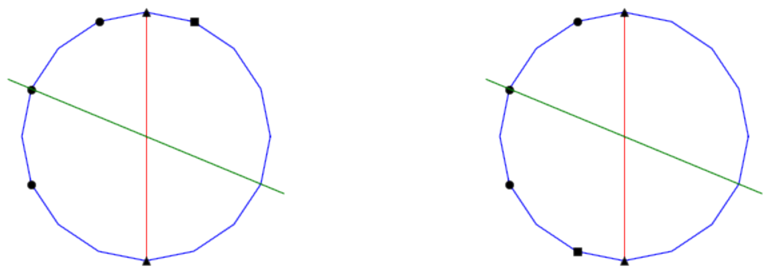

FIGURE 8. The set X_1 is conjugate to a subset of X_1' now represented by black dots in these figures. This new set is invariant under a different reflection allowing us to construct the third set of the triple.

9. Appendix: Golomb's Polynomial Method

For completeness we give an exposition of S. Golomb's approach (see e.g. [2]) to NCHP via generating functions.

9.1. **Factoring auto-correlation functions.** Let X and Y be a NCHP. We suppose that X consists of the points $x_0 = 0 < x_1 < \dots x_n = D$, where D is the diameter of X. Golomb associates to X the polynomial of degree D

$$r_X(t) := \sum_{x \in X} t^{d(x_0,x)}$$

It is easy to check that

$$r_X(t) r_X(1/t) = \sum_{(x,y) \in X^2} t^{x-y}.$$

Since X is a subset of $\mathbb{R}$, $d(x,y) = |x-y|$, and it follows immediately that the auto-correlation functions σ_X and σ_Y are equal if and only if

$$r_X(t) r_X(1/t) = r_Y(t) r_Y(1/t). \tag{7}$$

Golomb goes on to define

$$r_X^*(t) := t^D r_X(1/t) = \sum_{x \in X} t^{d(x_n,x)}$$

and note that X is invariant under reflection if and only if $r_X = r_X^*$. Further, the condition (7) is equivalent to

$$r_X r_X^* = r_Y r_Y^*. \tag{8}$$

Golomb then made a remarkable observation: the product $r_X(t) r_X^*(t)$ is a polynomial with integer coefficients and $\mathbb{Z}[t]$ is a unique factorisation domain so the equation (8) tells us that any irreducible factor of r_X is either a factor of r_Y or r_Y^*. Consequently, one has a decomposition of the polynomials into products

$$\begin{aligned} r_X &= \gcd(r_X, r_Y)(\gcd(r_X, r_Y^*)/\gcd(r_Y, r_Y^*)) \\ r_Y &= \gcd(r_X, r_Y)(\gcd(r_Y, r_X^*)/\gcd(r_X, r_X^*)). \end{aligned}$$

Note that $\gcd(r_X, r_Y^*) = \gcd(r_X^*, r_Y)$ so that

$$r_X = PQ,\ r_Y = PQ^* \tag{9}$$

where $P = \gcd(r_X, r_Y)$ and Q is the other factor.

Since $\mathbb{Z}[t_1, t_2 \dots t_n]$ is a UFD a similar result is true in higher dimensions.

9.2. **An example.** For example for the NCHP $\{0, 1, 4, 10, 12, 17\}$ and $\{0, 1, 8, 11, 13, 17\}$ cited in the introduction one has the decomposition

$$\begin{aligned} r_X(t) = 1 + t + t^4 + t^{10} + t^{12} + t^{17} &= (t^6 + t + 1)(t^{11} - t^5 + t^4 + 1) \\ r_Y(t) = 1 + t + t^8 + t^{11} + t^{13} + t^{17} &= (t^6 + t + 1)(1 - t^6 + t^7 + t^{11}) \end{aligned}$$

so that $\gcd(r_X, r_Y) = t^6 + t + 1$ and the other factors form a pair Q and Q^*.

Using the decomposition (9) for $n \geq 9$ with at least two factors greater than or equal to 3, one can construct NCHP with exactly n points quite easily for example:

$$\begin{aligned} (t^4 + t + 1)(t^7 + t^2 + 1) &= t^{11} + t^8 + t^7 + t^6 + t^4 + t^3 + t^2 + t + 1 \\ (t^4 + t^3 + 1)(t^7 + t^2 + 1) &= t^{11} + t^{10} + t^7 + t^6 + t^5 + t^4 + t^3 + t^2 + 1 \end{aligned}$$

9.3. **Underlying geometric construction.** There is an underlying simple geometric construction which generalises to $\mathbb{R}^n$ (see [1] and Figures 9 and 10 below). We say that a finite set $X \subset \mathbb{R}$, *asymmetric* iff it is not invariant under the inversion that swaps the minimum and maximum of X, namely

$$x \mapsto -x + \min(X) + \max(X).$$

Let $-X$ denote the set $\{-x \mid x \in X\}$ as usual and τ the translation $x \mapsto x+1$. Let $X_0 \subset]-\frac{1}{2}, \frac{1}{2}[$ be a set satisfying $\min(X_0) + \max(X_0) = 0$. Note that the diameter of X_0 is in fact $2\max(X_0)$. Then for any $X \subset \mathbb{Z}^+$ we can form the *Minkowski sums*

$$\begin{aligned} X_0 \oplus X &= \{a + b \mid (a, b) \in X_0 \times X\} \\ (-X_0) \oplus X &= \{-a + b \mid (a, b) \in X_0 \times X\}. \end{aligned}$$

Under the hypothesis on X_0

$$X_0 \oplus X = \bigsqcup_{n \in X} \tau^n(X_0), \tag{10}$$

we will say that each translate $\tau^n(X_0)$ is a *cluster* with center $n \in X$. Observe now that for any pair of clusters $\tau^n(X_0)$, $\tau^m(X_0)$ in $X_0 \oplus X$ the translation $\rho : x \mapsto -x+m+n$ maps them to the clusters to $\tau^m(-X_0)$, $\tau^n(-X_0)$ in $(-X_0) \oplus X$. Using this observation one can quite easily show that the auto-correlation functions of $X_0 \oplus X$ and $(-X_0) \oplus X$ are the same.

Moreover, provided both X_0 and X are asymmetric, these sets are not congruent. To see this note that, under our hypothesis on X_0,

$$m = \min((-X_0) \oplus X) = \min(X_0 \oplus X) = \min(X_0) + \min(X)$$

Let $n_0 = \min(X)$ and consider $\tau^{n_0}(X_0) \subset X_0 \oplus X$ and $\tau^{n_0}(-X_0) \subset (-X_0) \oplus X$. Then one has

$$\min(X_0) + \min(X) \in \tau^{n_0}(X_0) \cap \tau^{n_0}(-X_0)$$

and, since the diameter of X_0 is smaller than the minimal distance between distinct points of X, it follows that

$$B_m\left(\frac{1}{2}\right) \cap (\pm X_0 \oplus X) = \tau^{n_0}(\pm X_0).$$

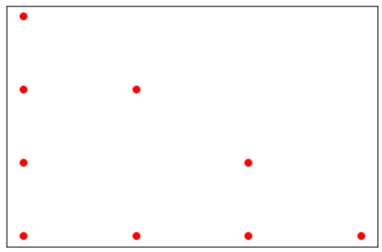

FIGURE 9. $X_0 \oplus X$.

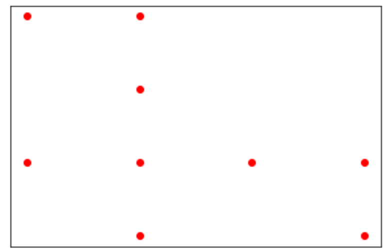

FIGURE 10. $(-X_0) \oplus X$.

So that if $(-X_0) \oplus X = X_0 \oplus X$ then $\tau^{n_0}(X_0) = \tau^{n_0}(-X_0)$ contradicting the hypothesis that X_0 was asymmetric. A similar argument can be used to show that $(-X_0) \oplus X$ and $X_0 \oplus X$ cannot be related by an inversion.

Figures 9 and 10 illustrate an example of homometric sets in $\mathbb{R}^2$ obtained from where $X_0 = \{(0,0),(1,0),(0,1)\}$ and $X = 2X_0$ by this process.

9.3.1. *Nine point configurations.* Without the hypothesis on the diameter of X_0 one cannot guarantee that $X_0 \oplus X$ decomposes as a disjoint union of translates as in (10) above. So, in the general case, one has to deal with so-called *multi sets*, that is families of points counted with multiplicities, and the most convenient way to do this appears to be via polynomials as above.

In order to satisfy the hypothesis that both X_0 and X are asymmetric they must both have at least three points and so the Minkowski sums each have at least nine points. The construction of homometric configurations with less than nine points requires much more care as can be seen from the factorisation in the previous section where one of the factors corresponds to a multiset where one point has a negative multiplicity

$$1 + t + t^4 + t^{10} + t^{12} + t^{17} = (t^6 + t + 1)(t^{11} - t^5 + t^4 + 1).$$

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