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On the Kähler form of the moduli space of once punctured tori

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A Riemann surface R of negative Euler characteristic has a unique hyperbolic metric. Provided R has finite area in this metric the Teichmüller space $T(R)$ of R will be a complex manifold. The complex structure of $T(R)$ is characterized by describing the holomorphic cotangent space at R. A natural identification exists of the holomorphic cotangent space and $Q(R)$, the space of holomorphic quadratic differentials on R. Consequently a Hermitian structure on $Q(R)$ naturally gives rise to one on $T(R)$. An example is the Petersson inner product. Given φ, $\psi \in Q(R)$ define

$$\langle \varphi, \psi \rangle = \int_R \varphi \bar{\psi} \lambda^{-2}$$

where λ^2 is the hyperbolic area element of R. The corresponding Hermitian structure on $T(R)$ is that of the Weil–Petersson metric. The metric is invariant with respect to the Teichmüller modular group and hence can be used to study the geometry of the moduli space of R. Ahlfors and Weil established that the metric is Kähler. We are concerned with the Kähler form ω of the metric.

A relationship exists between the geometry of ω and that of the vector fields derived from a construction of Fenchel–Nielsen. A Fenchel–Nielsen vector field $t(\alpha)$ on Teichmüller space is associated to each closed geodesic α of R. In the manuscript [10] the quantity $\omega(t(\alpha), t(\beta))$ is evaluated as the sum of the cosines of the intersection angles of α and β. It is also shown that the vector fields $t(\alpha)$ are Hamiltonian for ω; ω is invariant under the flow of $t(\alpha)$. The form ω and vector fields $t(\alpha)$ are the elements of a symplectic geometry for $T(R)$. The geometry is natural in the sense that ω is invariant with respect to the Teichmüller modular group. The quotient of $T(R)$ by the modular group is the classical moduli space of R. The Kähler form ω projects to the moduli space.

The simplest example of the above discussion is provided in the case of the once punctured torus. In the first section we describe natural global coordinates

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for the Teichmüller space $\mathcal{T}$ of the once punctured torus. These coordinates have no apparent relationship to the complex structure of $\mathcal{T}$. Teichmüller space in these coordinates is a simplex and the modular group $\mathcal{M}$ acts as a group of rational maps. An analysis of the action of $\mathcal{M}$ is given. The discussion is concluded with a description of a fundamental domain Δ. In the second section the Kähler form ω is calculated in the global coordinates and is found to be rational. Using the descriptions of ω and Δ the integral $\int_\Delta \omega$ is computed. It reduces to that of the dilogarithm. The final result is $\pi^2/6$, the area of the moduli space $\mathcal{T}/\mathcal{M}$.

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The Teichmüller space

We begin with an exposition of the Teichmüller theory of the once punctured torus. Our goal is to describe coordinates for the Teichmüller space, and to describe the action of the modular group. The material was in part previously considered by Keen, [3].

A once punctured torus is uniformized by a Fuchsian group Γ, $\Gamma \subset PSL(2;\mathbb{R})$. We shall use the following normalized form for the presentation of Γ. Hyperbolic transformations $A, B \in PSL(2;\mathbb{R})$ freely generate Γ with $ABA^{-1}B^{-1}$ parabolic; the repelling (resp. attracting) fixed point of A is 0 (resp. ∞) and the attracting fixed point of B is 1. In fact the group Γ can be lifted into $SL(2;\mathbb{R})$ such that $\operatorname{tr} A$, $\operatorname{tr} B$, $\operatorname{tr} AB$ become positive, where tr denotes the trace of a matrix. We shall consider Γ both as a subgroup of $SL(2;\mathbb{R})$ and of $PSL(2;\mathbb{R})$ without making the proper distinction. The quantities $x = \operatorname{tr} A$, $y = \operatorname{tr} B$, and $z = \operatorname{tr} AB$ uniquely characterize the above description of Γ. The transformation $ABA^{-1}B^{-1}$ is parabolic. An elementary argument shows that the commutator $ABA^{-1}B^{-1}$ has negative trace and consequently $\operatorname{tr} ABA^{-1}B^{-1} = -2$. The equation $\operatorname{tr} ABA^{-1}B^{-1} = -2$ is equivalent to the identity $x^2 + y^2 + z^2 = xyz$. This is the unique relation satisfied by the triple (x, y, z).

THEOREM (Fricke Klein [2], Keen [3]). *The Teichmüller space $\mathcal{T}$ of the once punctured torus is the sublocus of $x^2 + y^2 + z^2 = xyz$ satisfying $x, y, z > 2$.*

It will be necessary to consider two other coordinate systems for $\mathcal{T}$. We begin by introducing the invariants a, b and c where $a = x/yz$, $b = y/xz$ and $c = z/xy$. Teichmüller space is now the sublocus of $a + b + c = 1$ satisfying $a, b, c > 0$, a simplex. The third coordinate system will be introduced in the next section.

Distinct triples (x, y, z) and $(\tilde{x}, \tilde{y}, \tilde{z})$ may describe conjugate Fuchsian groups and thus isometric punctured tori. We wish to better understand this phenomenon. It is the direct consequence of the non-uniqueness of a choice of generators in the presentation for Γ. The automorphism group of Γ will be used to study the different choices of generators for Γ. Let G be the free group with generators A and B. The automorphism group $\mathrm{Aut}(G)$ of G has generators, σ, P and U where

$$\begin{matrix} \sigma(A) = A^{-1} & P(A) = B & & U(A) = AB \\ \sigma(B) = B & P(B) = A & \text{and} & U(B) = B \end{matrix},$$

[4]. A representation of $\mathrm{Aut}(G)$ in $GL(2; \mathbb{Z})$ is obtained by letting $\mathrm{Aut}(G)$ act on $G/[G, G] \approx \mathbb{Z} \oplus \mathbb{Z}$. Choosing the cosets of A and B as generators for $G/[G, G]$ we have under the representation

$$\sigma \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad P \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad U \to \begin{Bmatrix} 1 & 1 \\ 0 & 1 \end{Bmatrix}.$$

We shall be concerned with $\mathrm{Aut}^+(G)$ the preimage of $SL(2; \mathbb{Z}) \subset GL(2; \mathbb{Z})$. Denote by $\mathrm{Inn}(G)$ the inner automorphism group of G. The essential properties of the representation are given in the following theorem of Nielsen, [4].

THEOREM. *Let G be the free group on two generators. Then*

$$\mathrm{Out}^+(G) = \mathrm{Aut}^+(G)/\mathrm{Inn}(G) \approx SL(2; \mathbb{Z}).$$

In fact, by the representation

$$\sigma P \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad U \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

the classical generators of $SL(2; \mathbb{Z})$. By definition the Teichmüller modular group $\mathcal{M}$ for $\mathcal{T}$ is $\mathrm{Out}^+(G)$. The modular group $\mathcal{M}$ does not act effectively on $\mathcal{T}$; the kernel of the action is $\left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. Accordingly $PSL(2; \mathbb{Z})$ does act effectively on $\mathcal{T}$. It is this action that we wish to understand, in particular to describe a fundamental domain.

For clarification we note that the upper half plane H also provides a coordinate system for $\mathcal{T}$ where the action of $\mathcal{M} \approx SL(2; \mathbb{Z})$ is by fractional linear transformations. We shall describe the map from the (x, y, z) coordinates to the H coordinates. To the triple (x, y, z) the point $\tau \in H$ is determined as follows. First let Γ be the Fuchsian group associated to (x, y, z). Corresponding to the commutator sub-group $[\Gamma,\Gamma]\subset\Gamma$ is the abelian covering $H/[\Gamma,\Gamma]$ of H/Γ. The covering surface $H/[\Gamma,\Gamma]$ is planar and conformally equivalent to $\mathbb{C}-L$ for a Euclidean lattice L. A conformal map of $H/[\Gamma,\Gamma]$ to $\mathbb{C}-L$, equivariant with respect to $\Gamma/[\Gamma,\Gamma]$ and L, exists such that the cosets of A and B in $\Gamma/[\Gamma,\Gamma]$ are conjugated into the generators of L. The lattice L is normalized such that its generators are τ and 1 with $\tau\in H$. The quantity τ is the image of the triple (x, y, z).

We now wish to focus our attention on the principal congruence subgroup $\Gamma(2)$ of level 2 in $SL(2;\mathbb{Z})$, where

$$\Gamma(2)=\left\{C\in SL(2;\mathbb{Z})\mid C\equiv\begin{pmatrix}1&0\\0&1\end{pmatrix}\bmod 2\right\}.$$

Denote by $P\Gamma(2)$ the image of $\Gamma(2)$ in $PSL(2;\mathbb{Z})$. By an elementary argument the indices satisfy

$$[SL(2;\mathbb{Z}):\Gamma(2)]=[PSL(2;\mathbb{Z}):P\Gamma(2)]=6.$$

We wish to characterize the preimage of $P\Gamma(2)$ in $\mathrm{Aut}\,(G)$. Let $\mathcal{M}_2$ be the subgroup of $\mathrm{Aut}\,(G)$ generated by $\rho_1=\sigma U^2$, $\rho_2=\sigma$ and $\rho_3=P\sigma U^2P$. Under the representation of $\mathrm{Aut}\,(G)$ in $GL(2;\mathbb{Z})$

$$\rho_1\to\begin{pmatrix}-1&-2\\0&1\end{pmatrix}\qquad\rho_2\to\begin{pmatrix}-1&0\\0&1\end{pmatrix}\quad\text{and}\quad\rho_3\to\begin{pmatrix}1&0\\-2&-1\end{pmatrix}.$$

Denote by $\mathcal{M}_2^+$ the intersection $\mathcal{M}_2\cap\mathrm{Aut}^+(G)$. Now the representations of ρ_1, ρ_2 and ρ_3 each have negative determinant; consequently $\mathcal{M}_2^+\subset\mathcal{M}_2$ is the subgroup of words in ρ_1, ρ_2 and ρ_3 of even length. Under the representation of $\mathrm{Aut}\,(G)$ we claim that $\mathcal{M}_2^+\subset\Gamma(2)$, and that $\mathcal{M}_2^+$ surjects onto $P\Gamma(2)$. Indeed the representations of ρ_1, ρ_2 and ρ_3 are each congruent to $\begin{pmatrix}1&0\\0&1\end{pmatrix}\bmod 2$, the inclusion $\mathcal{M}_2^+\subset\Gamma(2)$ is immediate. The images of $\rho_2\rho_1$ and $\rho_2\rho_3$ are respectively $\begin{pmatrix}1&2\\0&1\end{pmatrix}$ and $\begin{pmatrix}-1&0\\-2&-1\end{pmatrix}$ the lifts of the generators of $P\Gamma(2)$. We shall establish below that $\mathcal{M}_2^+$ acts effectively and thus conclude that the natural map $\mathcal{M}_2^+\to P\Gamma(2)$ is a bijection.

First we shall consider the action of $\mathcal{M}_2$ in the (x, y, z) coordinates. We begin with the action of ρ_1,ρ_2 and ρ_3 on the generators A,B of Γ

$$\begin{array}{lll}\rho_1(A)=B^{-2}A^{-1} & \rho_2(A)=A^{-1} & \rho_3(A)=A\\ \rho_1(B)=B & \rho_2(B)=B & \rho_3(B)=A^{-2}B^{-1}\end{array}$$

There is natural induced action on the traces of the generators of Γ. Recalling that $x = \operatorname{tr} A$, $y = \operatorname{tr} B$, $z = \operatorname{tr} AB$ we have

$$\begin{array}{lll} \rho_1(x) = yz - x & \rho_2(x) = x & \rho_3(x) = x \\ \rho_1(y) = y & \rho_2(y) = y & \rho_3(y) = xz - y \\ \rho_1(z) = z & \rho_2(z) = xy - z & \rho_3(z) = z \end{array}$$

The following identity has been used: given C, $D \in SL(2; \mathbb{R})$ then $\operatorname{tr} C \operatorname{tr} D = \operatorname{tr} CD + \operatorname{tr} C^{-1}D$. The above triples satisfy $x^2 + y^2 + z^2 = xyz$, x, y, $z > 2$. For example consider the triple $(\rho_1(x), \rho_1(y), \rho_1(z))$. By the quadratic formula $2x = yz \pm (y^2z^2 - 4(y^2 + z^2))^{1/2} < 2yz$ hence $yz - x$ is positive. From the definition of $\operatorname{Aut}(G)$ there exists a unique normalized Fuchsian group $\tilde{\Gamma}$ with generators $\tilde{A}$, $\tilde{B}$ satisfying $|\operatorname{tr} \tilde{A}| = yz - x$, $|\operatorname{tr} \tilde{B}| = y$, $|\operatorname{tr} \tilde{A}\tilde{B}| = z$ and $\operatorname{tr} \tilde{A}$, $\operatorname{tr} \tilde{B}$, $\operatorname{tr} \tilde{A}\tilde{B} > 2$. Necessarily we have that $\operatorname{tr} \tilde{A} = yz - x$, $\operatorname{tr} \tilde{B} = y$ and $\operatorname{tr} \tilde{A}\tilde{B} = z$. The description of the $\mathcal{M}_2$ action is complete. We also give the action of ρ_1, ρ_2 and ρ_3 in the (a, b, c) coordinates

$$\begin{array}{lll} \rho_1(a) = 1 - a & \rho_2(a) = \dfrac{ca}{1-c} & \rho_3(a) = \dfrac{ba}{1-b} \\[2ex] \rho_1(b) = \dfrac{ab}{1-a} & \rho_2(b) = \dfrac{cb}{1-c} & \rho_3(b) = 1 - b \\[2ex] \rho_1(c) = \dfrac{ac}{1-a} & \rho_2(c) = 1 - c & \rho_3(c) = \dfrac{bc}{1-b} \end{array}$$

We shall now describe a fundamental domain for the action of $\mathcal{M}_2$ on $\mathcal{T}$. Consider the domain $\Delta \subset \mathcal{T}$, $\Delta = \{(a, b, c) \in \mathcal{T} \mid a, b, c \leq \frac{1}{2}\}$; Δ will be a fundamental domain for $\mathcal{M}_2$. Indeed this is an immediate consequence of the observation that the function $E(x, y, z) = x + y + z$ achieves a unique minimum on the orbit $\mathcal{M}_2(p)$, $p \in \mathcal{T}$ at the unique point of the orbit in Δ. We shall make the argument in several stages. E is the sum of positive traces from a fixed group Γ; the set of traces of elements of Γ is discrete. The minimum of E on the orbit $\mathcal{M}_2(p)$ exists. For the remaining arguments we require the following elementary formulas

$$\frac{E \circ \rho_1 - E}{2yz} = \frac{1}{2} - \frac{x}{yz} \qquad \frac{E \circ \rho_2 - E}{2xy} = \frac{1}{2} - \frac{z}{xy} \qquad \frac{E \circ \rho_3 - E}{2xz} = \frac{1}{2} - \frac{y}{xz} \tag{1}$$

Now if $q \in \mathcal{M}_2(p)$ represents a minimum then necessarily $E(\rho_j(q)) \geq E(q)$, $1 \leq j \leq 3$. In particular the formulas (1) show that the coordinates a,b,c of q are each bounded by $\frac{1}{2}$, hence $q \in \Delta$.

First we shall establish a convexity property of E along the orbit $\mathcal{M}_2(p)$. Let $w_n \cdots w_1$ be a reduced word in ρ_1, ρ_2 and ρ_3 (note that $\rho_j = \rho_j^{-1}$, $1 \le j \le 3$). Then we claim the finite sequence $E(s)$, $E(w_1(s)), \ldots, E(w_n \cdots w_1(s))$, $s \in \mathcal{T}$ is strictly convex. Consider the alternative: $n > 2$ and $r = w_j \cdots w_1(s)$, $j < n$ exist with $E(r) \ge E(w_{j-1} \cdots w_1(s)) = E(w_j(r))$ and $E(r) \ge E(w_{j+1}(r))$. Letting the coordinates of r be $(\tilde{a}, \tilde{b}, \tilde{c})$ then the inequalities $E(r) \ge E(w_j(r))$ and $E(r) \ge E(w_{j+1}(r))$ combined with (1) establish that two of the inequalities $\tilde{a} \ge \frac{1}{2}$, $\tilde{b} \ge \frac{1}{2}$ and $\tilde{c} \ge \frac{1}{2}$ necessarily hold, contradicting $(\tilde{a}, \tilde{b}, \tilde{c}) \in \mathcal{T}$. The convexity property is established. Consider now $q, r \in \Delta$ points of the orbit $\mathcal{M}_2(p)$. Using (1) we observe that $E(\rho_j(q)) \ge E(q)$, $E(\rho_j(r)) \ge E(r)$, $1 \le j \le 3$. Now these inequalities and the convexity of E imply that either $q = r$ or $\rho_k(q) = r$ for some k, $1 \le k \le 3$. In the latter case $E(r) = E(\rho_k(q)) \ge E(q)$ and $E(q) = E(\rho_k(r)) \ge E(r)$; the definitions of E and ρ_k show that actually $q = r$. In conclusion the minimum of E on an orbit $\mathcal{M}_2(p)$ occurs at its unique intersection with Δ.

We observe in closing that $\mathcal{M}_2$ acts effectively. Otherwise a nontrivial reduced word $w_n \cdots w_1$, $n > 2$ exists with $E(w_n \cdots w_1(q)) = E(q)$, $q \in \Delta$ contradicting the convexity of E. The subgroup $\mathcal{M}_2^+ \subset \mathcal{M}_2$ necessarily acts effectively; the natural map $\mathcal{M}_2^+ \to P\Gamma(2)$ is a bijection. As a consequence we have that the index $[PSL(2;\mathbb{Z})\colon \mathcal{M}_2^+]$ is 6.

The Kähler form

Our goal is to derive the expression for the Weil–Petersson Kähler form ω in the (a, b, c) coordinates and then intergrate ω over the fundamental domain Δ. We begin with the formula evaluating ω on the Fenchel Nielsen vector fields in terms of the hyperbolic geometry of closed geodesics on a Riemann surface. Then we proceed by a change of variables to obtain the desired formula. Finally the integral of ω over Δ is considered.

We consider only Riemann surfaces R with a hyperbolic, constant curvature -1, metric of finite area. A deformation of the metric R is defined by the following construction. Our point of view throughout is that a fixed topological surface underlies the conformal structure of R. Let γ be a simple closed geodesic. Cut R along γ, rotate one *side* of the cut relative to the other, and then glue the sides together in their new position. Perform this deformation continuously such that the distance between two points, one on each *side* of the cut, is measured by the time elapsed. A tangent vector of this deformation is an infinitesimal twist $t(\gamma)$. Our approach centers on the Fenchel–Nielsen tangents $t(\gamma)$. An important invariant of the hyperbolic metric on R is $l(\gamma)$, the length of the unique geodesic in the free homotopy class of γ.

In the manuscripts [8, 9, 10] the geometry of the quantities ω, $t(\alpha)$ and $l(\beta)$ is investigated. The fundamental formula is $\omega(t(\alpha),\ ) = -dl(\alpha)$. An immediate consequence is that the vector fields $t(\alpha)$ are Hamiltonian for the symplectic form ω. Indeed a symplectic geometry is associated to the quantities ω, $t(\alpha)$ and $l(\alpha)$. We shall only require the following two formulas. Denote by $\alpha \# \beta$ the intersection locus of the geodesics α and β then

$$\omega(t(\alpha), t(\beta)) = t(\alpha)l(\beta)$$

and

$$\omega(t(\alpha), t(\beta)) = \sum_{p \in \alpha \# \beta} \cos \theta_p$$

where θ_p is the intersection angle at p measured from α to β [9, 10].

Let $T(R)$ be the Teichmüller space of R and s its complex dimension. It is classical that free homotopy classes $\gamma_1, \ldots, \gamma_{2s}$ can be chosen such that the lengths $l(\gamma_j)$, $1 \le j \le 2s$ provide local real coordinates for $T(R)$ near R. We begin with an expression for ω in these coordinates.

LEMMA.

$$\omega = \sum_{j<k} W_{kj}\, dl_j \wedge dl_k$$

where the matrix (W_{jk}) *is the inverse of the matrix* $(t(\gamma_j)l(\gamma_k))$.

Proof. We abbreviate t_j for $t(\gamma_j)$ and l_j for $l(\gamma_j)$ and use repeated indices to indicate summation. By the chain rule $t_j l_m(\partial/\partial l_m) = t_j$, $1 \le j \le 2s$ and hence $\partial/\partial l_j = W_{jm}t_m$. Now we calculate

$$\begin{aligned}\omega &= \sum_{j<k} \omega\left(\frac{\partial}{\partial l_j}, \frac{\partial}{\partial l_k}\right) dl_j \wedge dl_k \\ &= \sum_{j<k} \omega(W_{jm}t_m, W_{kn}t_n)\, dl_j \wedge dl_k \\ &= \sum_{j<k} W_{jm}W_{kn}\omega(t_m, t_n)\, dl_j \wedge dl_k.\end{aligned}$$

Using the equation $\omega(t_m, t_n) = t_m l_n$ we proceed

$$\omega = \sum_{j<k} W_{jm} W_{kn} t_m l_n \, dl_j \wedge dl_k$$

$$= \sum_{j<k} W_{kj} \, dl_j \wedge dl_k,$$

and the calculation is complete.

The volume form dV of the Weil–Petersson metric by definition is $(1/s!)\omega^s = (1/s!)\omega \wedge \cdots \wedge \omega$. $\mathcal{T}$ has complex dimension 1; ω is the volume form of $\mathcal{T}$. Note that the Kähler metric on $\mathcal{T} \approx H$ is not complete and thus is not a multiple of the hyperbolic metric, [7]. We now argue that in the (x, y, z) coordinates

$$\omega = 4\frac{dx \wedge dy}{xy - 2z}$$

and then in the (a, b, c) coordinates

$$\omega = \frac{da \wedge db}{abc}.$$

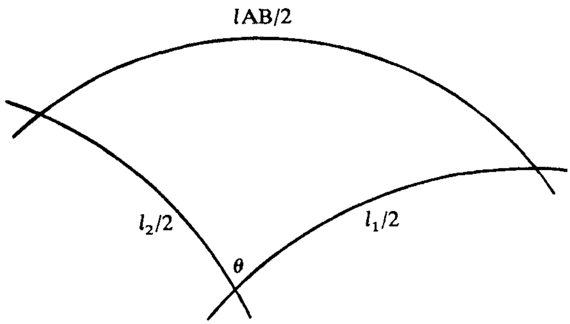
By definition if Γ represents a point of $\mathcal{T}$ with generators A, B then

$$x = \operatorname{tr} A = 2 \cosh l_1/2$$
$$y = \operatorname{tr} B = 2 \cosh l_2/2$$
$$z = \operatorname{tr} AB = 2 \cosh l_{AB}/2$$

where l_* is the length of the appropriate geodesic and x, y are local coordinates for $\mathcal{T}$ provided $xy - 2z \neq 0$. Applying the above lemma we have that

$$\omega = (t_1 l_2)^{-1} \, dl_1 \wedge dl_2$$

where $t_1 l_2 = \cos \theta$, and θ is measured *from* A to B. The following diagram indicates the geometry of the geodesics corresponding to A, B and AB



where the lengths are for the sides of the triangle, [7]. By the law of cosines

$$\cos\theta = \frac{\cosh l_1/2 \cosh l_2/2 - \cosh l_{AB}/2}{\sinh l_1/2 \sinh l_2/2}.$$

and thus

$$\omega = \frac{\sinh l_1/2 \sinh l_2/2}{\cosh l_1/2 \cosh l_2/2 - \cosh l_{AB}/2}\, dl_1 \wedge dl_2.$$

Now $dx = \sinh l_1/2\, dl_1$, $dy = \sinh l_2/2\, dl_2$ and on substituting we obtain

$$\omega = \frac{4\, dx \wedge dy}{xy - 2z}$$

the first expression.

The second expression is obtained by the rational change of variables. Beginning with the formulas $x^{-2} = bc$, $y^{-2} = ac$ and $z^{-2} = ab$ we have $-2\, dx/x = db/b + dc/c$ and $-2\, dy/y = da/a + dc/c$ thus

$$\frac{4\, dx \wedge dy}{xy} = \frac{db \wedge da}{ba} + \frac{db \wedge dc}{bc} + \frac{dc \wedge da}{ca}$$

or using $a + b + c = 1$

$$\frac{4\, dx \wedge dy}{xy} = \frac{a + b - c}{abc}\, da \wedge db.$$

Finally we obtain

$$\omega = \frac{a+b-c}{abc(1-2c)}\, da \wedge db = \frac{da \wedge db}{abc},$$

which is valid throughout $\mathcal{T}$ since ω is real analytic.

We are ready to consider the ω integral

$$\int_\Delta \omega = \iint\limits_\Delta \frac{da\, db}{ab(1-a-b)}.$$

Using the description of Δ we have

$$\begin{aligned}\int_\Delta \omega &= \int_0^{1/2}\int_{1/2-b}^{1/2} \frac{da\, db}{ab(1-a-b)} = \int_0^{1/2}\int_{1/2-b}^{1/2} \left(\frac{1}{a}+\frac{1}{1-b-a}\right)\frac{da\, db}{b(1-b)}\\ &= \int_0^{1/2} \frac{2}{b(1-b)} \log\frac{1}{1-2b}\, db\end{aligned}$$

and substituting $v = 1-2b$ yields the common dilogarithm integral

$$= -4\int_0^1 \frac{1}{1-v^2} \log v\, dv = \frac{\pi^2}{2}$$

[1].

We complete the calculation by considering the index of $\mathcal{M}_2^+ \subset PSL(2;\mathbb{Z})$. The form ω is $\mathcal{M}_2$ invariant and Δ is a fundamental domain for $\mathcal{M}_2$. Using the previously computed indices

$$\int_{\mathcal{T}/\mathcal{M}_2^+} \omega = [\mathcal{M}_2 : \mathcal{M}_2^+] \int_{\mathcal{T}/\mathcal{M}_2} \omega = \pi^2$$

$$\int_{\mathcal{T}/\mathcal{M}} \omega = \frac{1}{[PSL(2;\mathbb{Z}) : \mathcal{M}_2^+]} \int_{\mathcal{T}/\mathcal{M}_2^+} \omega = \frac{\pi^2}{6}$$

the area of moduli space for the once punctured torus.

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