THE DOUBLE LIFE OF PANTS

GREG MCSHANE

1. Introduction

This paper is about embedding Teichmeuller space of a bordered surface, for example of pants, in a bigger space –*the Thurston shear space*. The geometry of this space allows us to study deformations of certain geometric constructions in a convenient way.

Recall that a *pants* is a compact oriented triply connected surface together with a hyperbolic structure such that the boundary is totally geodesic. It is a planar surface and can be drawn (see figures) as a disc minus two disjoint subdiscs. The lengths of the three boundary components give a parameterization of the Teichmeuller space of pants [1]. When (at least) one of the boundary lengths is 0 the surface is still triply connected, non compact, though of finite volume. Such a surface is said to be *a degenerate pants*. A three punctured sphere is an example of degenerate pants.

There are two ways of constructing pants:

(1) as *the double* of a right angled hexagon along three of its six sides.
(2) as the *shear* of an ideal triangulation on a thrice punctured sphere.

The first point of view is the familiar one predominant in the litterature [9],[2] or [1] and even [**?**]. The second point of view, though less well known, has several advantages over the first not least of which is that the resulting pants comes with the additional structure of a distinguished maximal lamination.

Let M be a hyperbolic surface of genus g with $n > 1$ totally geodesic boundary components one of which is labelled γ. Following Mirzakhani's [7] let $\mathcal{T}_g(L_1 = \ell_\gamma, L_2 \dots L_n)$ denote the space of marked Riemann surfaces of genus g with n boundary components of lengths $L_1 = \ell_\gamma, L_2 \dots L_n > 0$. We introduce a deformation space, which we call the *enhanced Teichmueller space* , $\tilde{\mathcal{T}}_g(L_1 = \ell_\gamma, \dots L_n), L_i \neq 0$

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which is an *decorated* version of Mirzakhani's space $\mathcal{T}_g(L_1 = \ell_\gamma, \dots L_n)$. Formally the *decoration* consists of a choice of sign for each boundary component allowing us to consider signed (or oriented) lengths for oriented boundary curves. We identify this space with a subset of the space of shear coordinates of an ideal triangulation on a surface M of genus g with n punctures. this allows us to give a geometric interpretation of the sign and the signed length.

To define the sign we need the idea of peripheral system. Choose a maximal family of closed punctured discs $D_i, 1 \leq i \leq n\}$ in M such that for $i \neq j$ D_i, D_j meet in in exactly the basepoint of M; the boundary curve ∂D_i is freely homotopic to the i-th cusp. A *peripheral system* is the set of $\gamma_i^+ \in \pi_1(M, *)$ where γ_i^+ is a based loop representing ∂D_i with the boundary orientation from the inclusion in $M \setminus D_i$.

Recall that every complete hyperbolic structure on a punctured surface can be constructed by gluing ideal triangles along their edges. The gluing data is the set of *shear coordinates* [3] satisfying a linear system of constraints, namely that the sum of the shears round each of the peripheral loops is 0. When one modifies the shear data so that the sum of shears round γ_i^+ is no longer 0, one still obtains a surface homeomorphic to the punctured surface but the metric is incomplete. Its completion is a surface with boundary and the lengths of the boundary components are the absolute value of the sum of shears round the γ_i^+. The enhanced Teichmeuller space thus arises naturally on taking account the signs of these sums of shears.

More formally, the shear coordinates determine a developping map from the universal cover of the surface, viewed as a collection of ideal triangles modulo gluing maps, into $\mathbb{H}^2$. The associated holonomy representation

$$\mathrm{HOL} : \pi_1(M) \to \mathrm{PSL}(2, \mathbb{R}),$$

is not necessarily type preserving i.e. $\mathrm{HOL}(\gamma_i^+)$ may be not be parabolic. Let γ_i^+ be a peripheral loop such that $\mathrm{HOL}(\gamma_i^+)$ is hyperbolic. After conjugating if necessary we may assume that image of the developping map is contained in the right half space and that $x = 0$ is the axis of $\mathrm{HOL}(\gamma_i^+)$. The signed length ℓ_{γ_i} is defined by the equation

$$\mathrm{HOL}(\gamma_i^+)(z) = e^{\ell_{\gamma_i}} z$$

.

The simplest case of this construction, that of a pair of pants, is of particular interest and leads to deep results in moduli theory. The completion of $\tilde{\mathcal{T}}_0(\ell_\gamma, \ell_\alpha, \ell_\beta)$ can be identified with structures obtained by shearing along an ideal triangulation on a thrice punctured sphere.

2. Identities for pants

We develop this approach with a view to studying the so-called *McShane identity* and one of its generalizations. For surfaces of any genus $g \geq 1$ with a single puncture:

$$\sum_{\alpha,\beta} \frac{1}{1 + \exp(\frac{\ell_\alpha + \ell_\beta}{2})} = \frac{1}{2}. \tag{1}$$

where α, β are closed geodesics which bound an embedded (degenerate) pair of pants with the cusp. So that the RHS defines a constant function over the Teichmeuller space $\mathcal{T}_{g,1}$, $g \geq 1$.

We shall prove an enhanced version of Mirzakhani's theorem [7]:

Theorem 1. *For any surface in* $x \in \tilde{\mathcal{T}}_g(L_1 = \ell_\gamma, L_2 \dots L_n)$,

$$\sum_{\alpha,\beta} \mathcal{D}(\ell_\gamma, \ell_\alpha, \ell_\beta) + \sum_{\alpha,\gamma'} \mathcal{R}(\ell_\gamma, \ell_\alpha, \ell_{\gamma'}) = \ell_\gamma.$$

where

1. α, β *is a pair of closed non boundary geodesics which bound an embedded pair of pants with* γ.
2. γ' *is a boundary geodesic which bounds an embedded pair of pants with* α, γ.
3. $\mathcal{D}, \mathcal{R} : \mathbb{R}^3 \to \mathbb{R}$ *are real analytic functions.*

Explicit expressions for gap functions

The first aim of the paper is to define and calculate the functions $\mathcal{D}, \mathcal{R}$, the so called *gap functions*, using shear coordinates. Mirzakhani [7] (see also [14]) gives explicit expressions for the gap functions:

Lemma 2.

$$\mathcal{D}(x, y, z) = 2 \log \left(\frac{\exp(\frac{x}{2}) + \exp(\frac{y+z}{2})}{\exp(-\frac{x}{2}) + \exp(\frac{y+z}{2})} \right)$$

$$\mathcal{R}(x, y, z) := x - \log \left(\frac{\cosh(\frac{y}{2}) + \cosh(\frac{x+z}{2})}{\cosh(\frac{y}{2}) + \cosh(\frac{x-z}{2})} \right).$$

These functions appear as factors in unfoldings of integrals over moduli space and as such are of fundamental importance in the theory of intersection numbers of Chern classes over these spaces [6], compare [**?**].

The arguments in the first gap function are essentially $x, y + z$ and, in some sense, this observation is really the starting point for this paper. This as well as the fact that $\mathcal{D}$ is odd in x i.e.

$$\mathcal{D}(x, y, z) = -\mathcal{D}(-x, y, z) \tag{2}$$

and the identity

$$\mathcal{R}(x, y, z) = \frac{1}{2}(\mathcal{D}(x, y, z) + \mathcal{D}(x, y, -z)) \tag{3}$$

appear naturally, and without the use of any hyperbolic trigonometry, from our construction. The essential step in proving Lemma 2 involves calculations using only affine transformations parameterized by shear coordinates.

3. A localization of the Birman-Series set

Let M be a convex surface of finite area; we suppose (see [2]) that M is represented as convex core of $\mathbb{H}^2/\Gamma$ for $\Gamma = \rho(\pi_1)$ a fuchsian representation of $\pi_1(M)$. Recall that the *Birman Series set* is the set of all complete simple geodesics on M. in [5] we studied so-called *gaps* in the Birman Series set in a neighborhood of a cusp. The second part of the paper is concerned with giving a formalism for gap that works for surfaces with cusps as well as surfaces with boundary. To do this we translate the ideas of [5] into the language of fuchsian groups acting on $\partial\mathbb{H}^2 = S^1$. The limit set of Γ, $\Lambda \subset \partial\mathbb{H}^2$ is the (unique) minimal closed Γ invariant subset pof $\partial\mathbb{H}^2$. We define and a "localization", $\hat{X}_{ess}(p)$,of the Birman Series set at a peripheral point p in the limit set; the reader should think of p as being the positive fixed point of a $\mathrm{HOL}(\gamma_i^+)$ above.

The set of recurrent geodesics on M can be viewed as the quotient of the complement of the diagonal in $\Lambda \times \Lambda$ by the diagonal action of Γ. Following Manin we use the notation $\{a, b\}, a, b \in \partial\mathbb{H}^2$ to represent the geodesic joining a to b and oriented/directed in this direction.

An element $h \in \Gamma$ is *peripheral* if it is parabolic or if it is hyperbolic and its axis covers a boundary component. We say that $p \in \Lambda$ is a *peripheral point* iff p is fixed by a peripheral element and write Λ^{per} for the set of peripheral points. For $p \in \Lambda^{per}$ let $\hat{X}(p)$ be the set of $x \in \Lambda$ such that the geodesic $\hat{\gamma}_x = \{p, x\}$ covers a simple geodesic γ_x on the surface. We write

$$X(p) := \hat{X}(p)/\mathrm{stab}(p).$$

where $\mathrm{stab}(p) < \Gamma$ be the stabiliser of p. Observe that, when p is parabolic then one may identify $x \in X(p)/\mathrm{stab}(p)$ with the $\gamma_x \cap \partial\mathcal{H} \in E \cap \partial\mathcal{H}$ – where $\partial\mathcal{H}$ is a small horocycle based at the corresponding cusp on M and E is the set of ends of simple geodesics going up the cusp p (compare [5]).

Let us now say precisely what it means, in terms of points of $\partial\mathbb{H}^2$, for one geodesic to spiral to another on M. Let $\hat{\gamma} = \{c^-, c^+\} \subset \mathbb{H}^2$ cover a directed geodesic $\gamma \subset M$. A geodesic δ *spirals* to γ iff there is a lift $\hat{\delta}$ with an endpoint in the set $\Gamma.\{c^+\}$. Note that if γ is closed then γ it is contained in the closure of δ. Note also that if γ is an oriented boundary geodesic then the set of simple geodesics which spiral to γ is just the set of γ_x with $x \in X(p)$ for $p = c^+$.

By the theory of laminations [**?**] $\hat{X}(p)$ is closed and, by the Birman-Series Theorem, it is nowhere dense so it is a Cantor set union countably many isolated points. A point x of a Cantor set in $\mathbb{R}$ is a *boundary point* iff it is in the closure of an open interval in the complement of $X_{ess}(p)$ and is a *deep point* otherwise. We write $\Omega(\gamma_x)$ for the minimal lamination in the closure of γ_x and we say that γ_x *spirals to* $\Omega(\gamma_x)$. The lamination can be one of three things:

- $\Omega(\gamma_x)$ is γ_x a single geodesic with both ends in cusps iff x too is a parabolic point.
- $\Omega(\gamma_x)$ is a single simple closed geodesic iff x is a hyperbolic point.
- $\Omega(\gamma_x)$ something much more complicated $-$*an arational lamination* $-$ if x is neither parabolic nor hyperbolic.

In [5] it was shown that these types correspond to respectively the isolated, boundary and deep points of $X(p)$ (Theorem 3 below).

Let

$$\hat{X}_{ess}(p) := \hat{X}(p) \setminus \Lambda^{per}.$$

With these definitions, Theorem 4 of [5] can be restated:

Theorem 3. *Let $N = \mathbb{H}^2/\Gamma$ be a surface (convex, finite area without boundary) with a cusp. Let p be a peripheral point and $x \in \hat{X}(p)$:*

(1) *x is isolated iff x is a peripheral point i.e. $x \in \Lambda^{per}$, so that $\hat{X}_{ess}(p)$ is a Cantor set.*
(2) *x is a boundary point of $\hat{X}_{ess}(p)$ iff it is the fixed point of a (non trivial) non peripheral element of Γ.*
(3) *Every gap contains a peripheral point.*

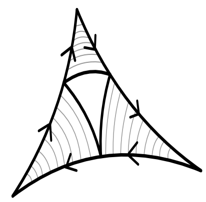

FIGURE 1. The partial foliation and the edge orientations on an ideal triangle.

Definition A *gap* is a component of the complement of $\hat{X}_{ess}(p)$. Note that this is not quite the same definition as in [5] but it is consistent and we use it throughout this article. The terms in Theorem 1 are in fact the sizes of gaps and part (3) above says essentially that the gaps are "indexed" by peripheral points of $\hat{X}(p)$.

We shall use this version of the theorem to prove the key lemma, namely:

Lemma 4. *Let Γ be a fuchsian representation of $\pi_1(M)$. Every gap in $\hat{X}_{ess}(p)$ contains a peripheral point $p' \in \Lambda^{per}$.*

Each gap in $\hat{X}_{ess}(p)$ gives rise to a gap in $\hat{X}_{ess}(p)/\mathrm{stab}(p) = X_{ess}(p)$ in the obvious way.

Corollary 5. *Every gap in $X_{ess}(p)$ arises from the gap construction in pants (Lemma 7).*

The identity in Theorem 1 follows immediately from Corollary 5 and Lemma 2 plus the Birman-Series Theorem i.e. the Lesbegue measure of $X(p)$ is zero so that the sum of sizes of gaps is the (signed) length of the boundary γ.

4. The geometry of an ideal triangle

Everything hinges on understanding the geometry of an ideal triangle so, for completeness, we recall several definitions. An *ideal triangle* is the convex hull of three distinct points at infinity in $\mathbb{H}^2$. There is only one such triangle up to isometry and its area is π. When we say *an ideal triangle in a surface* we always mean an embedded ideal triangle in the surface.

We choose an orientation for the edges of ideal triangle as in the Figure 1 and call this the *canonical orientation*.

Each side of an ideal triangle contains an unambiguously defined *midpoint* sometimes called a *tick mark*. The midpoint of an edge can be characterised by saying that it is the fixed point of the unique involution of the triangle that swaps the endpoints of the edge.

There is a natural partial foliation of an ideal triangle by horocyclic segments based at the vertices as per the diagram. A *corner or prong* of an ideal triangulation is a sub region containing a single ideal vertex and bounded by a leaf of this foliation. A calculation shows that the length of a leaf is just the area of the corner which it bounds.

Finally, there are three *osculating leaves* with endpoints at the tick marks/midpoints. Another calculation shows that length of a leaf at height h above an osculating leaf is $\exp(-h)$.

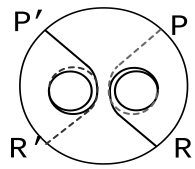

FIGURE 2. The four points Q,Q',R,R'.

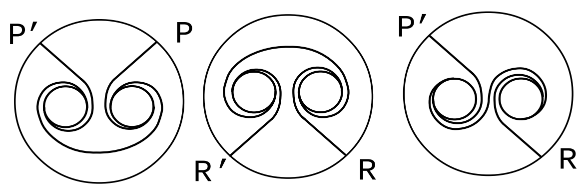

FIGURE 3. Grimace d'enfants: two smilers and a twister.

5. Geodesics on pants

We start by reviewing the geometric definition of the gap functions $\mathcal{R}, \mathcal{D}$. Label the boundary curves of a pants α, β, γ. There are four special points $Q, Q', R, R' \in \gamma$ (Figure 4) these are the initial points of simple geodesics which meet γ perpendicularly and spiral to α or β. The function $\mathcal{R}$ is the distance between the points R, Q (see Figure 3) going *anti-clockwise*. Mirzakhani [7] views this quantity as $\ell_\gamma-$ size of projection of β onto γ and defines

$$\mathcal{D}(x, y, z) := \mathcal{R}(x, y, z) + \mathcal{R}(x, z, y) - x.$$

The function $\mathcal{D}$ is in fact the sum of the lengths of the distances (going anti-clockwise) between P, P' and R', R. The intervals P, P' and R', R turn out to be gaps; recall that a *gap* is a portion of the surface that meets no complete simple geodesic. These gaps have the same size and are swapped by an involution of the pants.

Orientation on the boundary

Let P be an oriented pair of pants and $\alpha^+, \beta^+, \gamma^+ \in \pi_1(P, *)$ a peripheral system for P We think of P as being the quotient of $\mathbb{H}^2$ by a fuchsian represenation $\rho : \pi_1 \to \Gamma$. Let a^-, a^+ denote respectively the attracting and repelling fixed points of $\rho(\alpha^+)$ likewise b^-, b^+ the attracting and repelling fixed points of $\rho(\beta^+)$ and c^-, c^+ for $\rho(\gamma^+)$ The attracting and repelling fixed points of $\rho(\alpha^+), \rho(\beta^+), \rho(\gamma^+)$ alternate on $\mathbb{H}^2$ as in Figure 4 on the left.

Spinning, decorated pants

Let γ^+ be an oriented boundary. Take any complete geodesic with an endpoint on γ^+, push the endpoint along γ^+ in the preferred direction to get a sequence of geodesics converging towards a complete geodesic *asymptotic* to γ^+. When one carries out this procedure with the geodesics of the smiler or the twister one obtains a configuration the closure of which is a maximal lamination (see figure 4) with exactly six leaves – the three boundary geodesics plus three leaves that spiral between the boundary components.

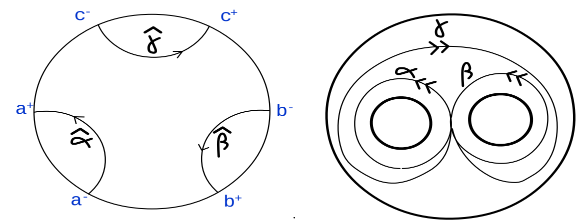

FIGURE 4

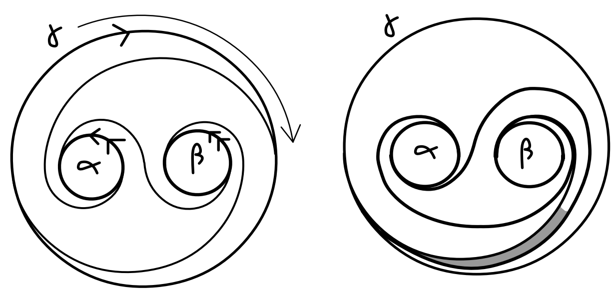

FIGURE 5. Spinning a twister and a smiler. The thin corner is the "bottom lip" of the smiler.

Definition Consider the set of oriented pants P with a *decoration* that consists of a choice of maximal lamination $\lambda \subset P$ such that no leaf of λ separates. The coordinates on this space are signed boundary lengths $(\ell_{\gamma_1}, \ell_{\gamma_2}, \ell_{\gamma_3})$ where the sign of ℓ_{γ_i} is positive iff there is a leaf of λ spiralling to γ_i^+. The *enhanced Teichmeuller space of pants* is the set of (P, λ) with these coordinates. The *completed enhanced Teichmeuller space of pants* is the set of pairs (P, λ) where P is now allowed to be a degenerate pants i.e. with a cusp instead of a geodesic boundary component γ_i; the usual convention applies in this case that $\ell_{\gamma_i} = 0$.

Example: The twister and the smiler in Figure 4 live on a pants P with boundary lengths $\mid \ell_\alpha \mid, \mid \ell_\beta \mid, \mid \ell_\gamma \mid$. The pair ($P$, twister) has coordinates $(\mid \ell_\alpha \mid, \mid \ell_\beta \mid, \mid \ell_\gamma \mid)$ whilst (P, smiler) has coordinates $(- \mid \ell_\alpha \mid, \mid \ell_\beta \mid, \mid \ell_\gamma \mid)$ in the enhanced Teichmueller space.

6. SHEAR COORDONATES FOR THE THREE PUNCTURED SPHERE

We now construct pairs of pants from a pair of ideal triangles using Thurston's shear construction.

Let T, T' be a pair of ideal triangles together with an isometric gluing map $\mathcal{I}$ from the edges of T to the edges of T' so that $P := (T \cup T')/\mathcal{I}$ is a surface homeomorphic to the interior of a pants. Label the midpoints/tickmarks of T A, B, C and the midpoints of T' A', B', C' such that $\mathcal{I}(A)$ is mapped into the edge containing A' etc. Give each edge of T' its canonical orientation and let a be the signed distance from A' to A, b be the signed distance from B' to B and c be the signed distance from C' to C with respect to this orientation. The numbers $(a, b, c) \in \mathbb{R}^3$ are called *shear coordinates*. Since the restriction of $\mathcal{I}$ to each edge of T is an isometry the

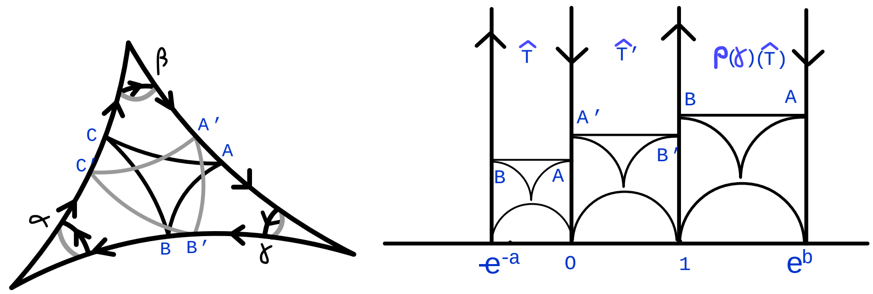

FIGURE 6. On the left a three punctured sphere with an ideal triangulation. On the right the developping round γ in $\mathbb{H}^2$.

shear coordinates parameterize all possible gluing maps $\mathcal{I}$. Define the *shear space* to be the set of all surfaces obtained by varying a, b, c.

Note that, since P is obtained from $T \cup T'$ we have extra structure coming from the canonical map $T \hookrightarrow (T \cup T') \to P$, we interpret this as the choice of a maximal lamination $\lambda =$ (the closure of the edges of T) on the completion, $\bar{P}$, of P. Thus it is natural to try and identify the shear space and the space of decorated pants introduced above. We do this by constructing a diffeomorphism between these spaces – a change of coordinate map.

To determine this change of coordinate map we want to calculate the image of the three peripheral loops $\alpha^+, \beta^+, \gamma^+ \in \pi_1$ under the holonomy representation HOL.

Let us see how the developping map from the universal cover $\hat{P}$ of $P = (T \cup T')/\mathcal{I}$ into $\mathbb{H}^2$ determines the holonomy $\mathrm{HOL}(\gamma^+)$ round γ^+. We identify T with the ideal triangle $\hat{T} = 0, -e^a, \infty$ and T' with $\hat{T}'$ the triangle $1, 0, \infty$ so that $\hat{T} \cup \hat{T}'$ is the image under the developping map of a fundamental domain for the action of the deck transformations on $\hat{P}$; a quick calculation shows that the shear along $\{\infty, 0\}$ is indeed a. The geodesic $1, \infty$ is the intersection of $\hat{T}'$ with another ideal triangle namely $\mathrm{HOL}(\gamma^+)(\hat{T})$. On the other hand, the shear coordinate along the corresponding edge in $(T \cup T')/\mathcal{I}$ is b so that $\mathrm{HOL}(\gamma)(\hat{T})$ has vertices $1, 1 + e^b, \infty$. Considering the vertices of $\hat{T}$ one sees that the holonomy satisfies

$$-e^a \mapsto 1, 0 \mapsto 1 + e^b, \infty \mapsto \infty,$$

and, since it is an isometry of $\mathbb{H}^2$, this means

$$\mathrm{HOL}(\gamma^+)(z) = e^{(a+b)} z + (e^b + 1), \; \forall z \in \mathbb{H}^2 \cup \partial \mathbb{H}^2.$$

Suppose $a + b \neq 0$ so that $\mathrm{HOL}(\gamma^+)$ fixes ∞ and $\eta = \frac{1+e^b}{1-e^{a+b}}$. We now determine whether the edges of T spiral to γ^+ or γ^-. From the discussion of spiralling in Section 3 it suffices to check if ∞ is respectively the attracting or repelling fixed point of $\mathrm{HOL}(\gamma^+)$. If $a + b > 0$ then $\eta < 0$ and the image of the developping map is contained in the half space $x > \eta$, the geodesic ∞, η covers a boundary component of the completion $\bar{P}$ of length $a + b$. Moreover ∞ is the attracting fixed point of $\mathrm{HOL}(\gamma^+)$ and so the edges of T spiral to γ^+. Similarly, if $a + b < 0$ $\eta > 0$ and the image of the developping map is contained in the half space $x < \eta$, the geodesic ∞, η covers a boundary component of the completion of $(T \cup T')/\mathcal{I}$ of length $\mid a + b \mid$.

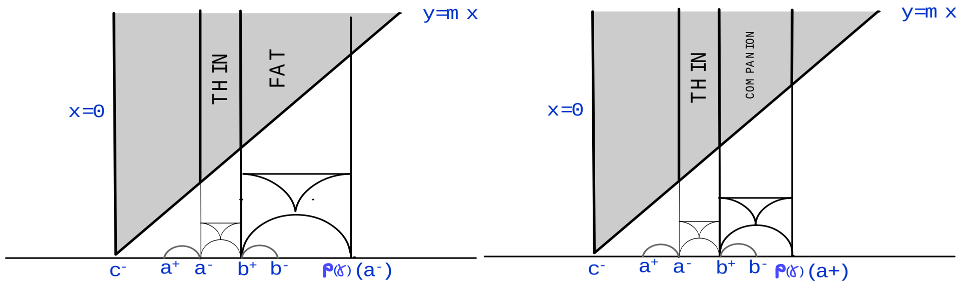

FIGURE 7. The normalized lifts used to calculate $\mathcal{G}, \mathcal{H}$.

Moreover ∞ is the repelling fixed point of $\mathrm{HOL}(\gamma^+)$ and so the edges of T spiral to γ^-. In either case one concludes that $a + b = \ell_\gamma$ as required.

When $a + b = 0$ the holonomy $\mathrm{HOL}(\gamma^+)$ is clearly parabolic, $\ell_\gamma = 0$ i and the geodesics go up the cusp.

By symmetry of the gluing map $\mathcal{I}$ one obtains:

Shear equations

$$\begin{aligned} b + c &= \ell_\alpha \\ a + c &= \ell_\beta \\ a + b &= \ell_\gamma \end{aligned}$$

This proves:

Lemma 6. ***Change of coordonates*** *The map*

$$(a, b, c) \mapsto (b + c, a + c, a + b).$$

is a diffeomorphism between the shear space of the thrice punctured sphere and the completed enhanced Teichmueller space of pants.

In fact, as we shall see below, only the value of $a = \frac{1}{2}(-\ell_\alpha + \ell_\beta + \ell_\gamma)$, where $\ell_.$ denotes signed length, is important in the calculation of the gap functions.

7. Pants and gaps

Both the smiler and the twister have two corners winding round γ. For the smiler one of these corners, which we call the *thin corner* (see Figure 4 and Figure 7), is special. Given a twister and a smiler, on the same pants of course, they have exactly one geodesic in common and the twister has a corner disjoint from the thin corner in the smiler. We call this latter corner the *companion of the thin corner.*

Now suppose that the pants P is a surface M with a disingushed peripheral loop $\gamma^+ \subset P$. Let $X_{ess}(p)$ be as in the introduction for p a choice of endpoint c^-, c^+ of a lift of γ^+. The following observation is essentially made in [5]:

Gap principle: every geodesic p, x, $x \in X_{ess}(p)$ which does not spiral to either α or β crosses either α or β.

Lemma 7. ***Gaps exist***

(1) *If neither α nor β is a boundary geodesic then the thin corner in a smiler is a gap.*

(2) *If β is a boundary geodesic then there is a gap containing the thin corner of the smiler and its companion in the twister.*

Proof:

$\square$

8. Gap functions

In this section we define and determine explicitly the functions $\mathcal{G}, \mathcal{H}$ appearing in Theorem 1;' $\mathcal{G}$ is none other than the "width" of the thin corner and $\mathcal{H}$ the "width" of a companion corner as defined above. By the preceding lemma $\mathcal{G}$ and $\mathcal{G}+\mathcal{H}$ are the signed "widths" of gaps in $X_{ess}(p)$.

We continue use the same notation (Section 5, Figure 4) for peripheral elements and endpoints. In order to calculate these functions in terms of signed lengths we normalize the covering map $\pi_M : \mathbb{H}^2 \to M$ where $M = \mathbb{H}^2/\Gamma$. By covering theory picking a normalization is equivalent to conjugating the fuchsian group Γ. We conjugate Γ so that $z \mapsto e^{\ell_\gamma} z$ is the holonomy round the boundary component γ^+. It follows that the thin corner lifts into an ideal triangle the sides of which are vertical lines $x = a^-$ and $x = b^+$. The sides of the companion corner lifts into an ideal triangle the sides of which are $x = b^+$ and $x = e^{\ell_\gamma} a^+$.

Observation:

Observe that one can conjugate Γ by any element of $\langle z \mapsto e^s z,\ s \in \mathbb{R}\rangle$ whilst preserving this normalization and that dt/t is an invariant 1-form for this group of transformations.

If $z \in X(p) \subset \Lambda$, $p = \infty$ then $z < e^{\ell_\gamma} z$ and we may write

$$\ell_\gamma = \int_z^{e^{\ell_\gamma} z} \frac{dt}{t}.$$

One views this as the RHS of the identity in Theorem 1.

We define

$$\mathcal{G} := \int_{a^-}^{b^+} \frac{dt}{t}$$

and

$$\mathcal{H} := \int_{b^+}^{e^{\ell_\gamma} a^+} \frac{dt}{t}.$$

Remarks

(1) One can show that $\mathcal{G}$ is the length of the arc between P and P' and $\mathcal{H}$ the length of the arc between P' and R. The proof is by choosing the sequences of geodesics in the spinning construction carefully and is left to the reader. Moreover, these are related to Mirzikhani's functions by:

$$\mathcal{D} = 2\mathcal{G},\ \mathcal{R} = \mathcal{G} + \mathcal{H}.$$

(2) One can give an interpretation of $\mathcal{G}$ in terms of the intrinsic geometry of M as follows. For $m > 0$ sufficiently big the cone $y > mx, x > 0$ projects to an embedded annulus with boundary components γ and another (simple) curve γ_m. Moreover any simple geodesic which enters such annulus never leaves and in particular the intersection of each of the geodesics bounding

the thin corner and γ_m consists of a single point. On the other hand the length of γ_m is

$$\int_z^{e^{\ell_\gamma} z} \left(1 + \frac{1}{m^2}\right) \frac{dt}{t} = \left(1 + \frac{1}{m^2}\right) \times \ell_\gamma$$

and the distance between the two intersection points of the edges of the thin corner (measured along γ_m) is $\mathcal{G} \times (1 + \frac{1}{m^2})$. Similarly one can give a geometric interpretation of $\mathcal{H}$.

Using our philosophy of viewing the smiler and twister as solutions of the shear equations, one sees that

$$\mathcal{H}(x, y, z) = \mathcal{G}(x, y, -z),$$

thus it suffices to calculate b^+ to determine the gap functions. In the following sections we use shear coordinates on the thrice punctured sphere to do this.

9. Calculating the gap function $\mathcal{D}$: Proof of Lemma 2

The cusped case

The case when the total shear round the peripheral loop is 0, so that $\ell(\gamma) = 0$, is perhaps easiest to understand. We can actually carry out the calculation without lifting to $\mathbb{H}^2$. Recall (see [5]) that the cusp region of area 2 is foliated by concentric horocylces. The corners of embedded ideal triangles map into the cusp region in such a way that the leaves of the partial foliation map into leaves of the horocylic foliation.

Let's calculate $\mathcal{G}(0, \ell_\alpha, \ell_\beta)$, compare [5]. The osculating leaf bounding the fat corner in the smiler is contained in a leaf L of the horocyclic foliation. We need to calculate the proportion of L in the thin corner. If the portion of L in the fat corner has length 1 then the portion in the thin corner is a leaf of the foliation of the ideal triangle at height a above the osculating leaf bounding the corner. Thus this portion has length e^{-a} and we can write

$$\frac{\text{thin corner}}{\text{fat corner}} = e^{-a}.$$

The total length of L is $1 + e^{-a} < 2$, so fraction of the horocycle in the gap is

$$\mathcal{D}(0, \ell_\alpha, \ell_\beta) := \frac{\exp(-a)}{1 + \exp(-a)} = \frac{1}{1 + \exp(a)} = \frac{1}{1 + \exp(\frac{|\ell_\alpha| + |\ell_\beta|}{2})}.$$

Similarly we can calculate $\mathcal{H}(0, \ell_\alpha, \ell_\beta)$ by comparing the fraction of an embedded horocycle in the companion corner going up the cusp. We see that there is an embedded horocycle of total length $1 + \exp(-a)$ such that the length of the subleaf inside our corner is again $\exp(-a)$ where $a = \frac{1}{2}(\mid \ell_\alpha \mid - \mid \ell_\beta \mid)$ since a smiler and a twister have the same coordinates except for a sign change. Hence:

$$\mathcal{H}(0, \ell_\alpha, \ell_\beta) := \frac{1}{1 + \exp(\frac{|\ell_\alpha| - |\ell_\beta|}{2})}.$$

The non cusped case

When the total shear round the peripheral loop representing γ is non zero one can again find the gap function using shear coordinates. Lift to $\mathbb{H}^2$ as in section 4 above and let a^-, b^+ be as defined previously. Using the shear coordinate argument as in the cusped case

$$\frac{\text{thin corner}}{\text{fat corner}} = \frac{b^+ - a^-}{e^{\ell_\gamma} a^- - b^+} = e^{-a},$$

where $a = \frac{1}{2}(-\ell_\alpha + \ell_\beta + \ell_\gamma)$. After conjugating Γ if necessary so that $a^- = \frac{e^{-\ell_\gamma}}{1-e^{-\ell_\gamma}}$ and using the above equation to solve for b^+

$$b^+ = \frac{e^{-\ell_\gamma}}{1 - e^{-\ell_\gamma}} + \frac{1}{1 + e^a}.$$

Now

$$\mathcal{G}(\ell_\gamma, \ell_\alpha, \ell_\beta) := \int_{a^-}^{b^+} \frac{dt}{t} = \log\left(1 + \frac{1 - e^{-\ell_\gamma}}{e^{-\ell_\gamma}(1 + e^a)}\right) - \log(1) = \log\left(\frac{1 + e^{-\ell_\gamma} e^a}{e^{-\ell_\gamma} + e^{-\ell_\gamma} e^a}\right).$$

Applying our principle as in the cusped case above for $\mathcal{H}$

$$\mathcal{H}(\ell_\gamma, \ell_\alpha, \ell_\beta) = \log\left(\frac{1 + e^{-\ell_\gamma} e^a}{e^{-\ell_\gamma} + e^{-\ell_\gamma} e^a}\right) = \mathcal{G}(\ell_\gamma, \ell_\alpha, -\ell_\beta),$$

where $a = \frac{1}{2}(\ell_\alpha - \ell_\beta + \ell_\gamma)$. It is left to the reader to check that the expression for $\mathcal{R} = \mathcal{H} + \mathcal{G}$ agrees with that given in the introduction.

10. Conjugation between surfaces groups.

It is a theorem of classical Teichmeuller theory that given a pair of compact Riemann surfaces (without boundary) of the same genus A, B there is a quasi conformal homeomorphism $\phi : A \to B$. Lifting to the universal covers one obtains a quasiconformal homeomorphism $\hat{\phi}$ of the unit disc and this induces a quasisymmetric homeomorphism $\Phi : \partial\mathbb{H}^2 \to \partial\mathbb{H}^2$ – the *boundary map.* In fact this map *conjugates* the actions of ρ_A, ρ_B on S^1 that is :

$$\Phi \circ \rho_A(\alpha) \circ \Phi^{-1} = \rho_B(\alpha), \forall \alpha \in \pi_1.$$

Remarks

1. The density of fixed points of hyperbolic elements in the limit set Λ ($= S^1$) means that Φ is unique.
2. In a more modern treatment one views the limit set as the *Gromov boundary* of the group (see e.g. Mitra [8]) ansd one uses only that $\hat{\phi}$ is a bi-Lipschitz or even more generally a quasi-isometry. In fact any Lipschitz map $\hat{\phi}$: $\mathbb{H}^2 \to \mathbb{H}^2$ admits a unique extension to $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ though this might not be injective.

Non compact For hyperbolic surfaces A, B of genus g with $n > 0$ totally geodesic boundary components one can find Φ that conjugates the actions of ρ_A, ρ_B.

For example one can take doubles and use the result for the compact case. Note that in this case Φ is not unique.

Pinched surfaces, semiconjugacy.

Let M a surface with $\gamma_i^+ \in \pi_1(M),\ 1 \leq i \leq n$ a peripheral system. The results of our previous paper [5] apply to surfaces N with cusps and we now associate such a surface to M by *pinching off* the boundary components to cusps.

More precisely choose an orientation preserving diffeomorphism $f : M \to N$ where N be a Riemann surface with n punctures that takes the peripheral system

$\{\gamma_i^+\}$ to a peripheral system i.e. $f_*(\gamma_i^+)$ is a loop round a puncture. If ρ_N is a representation of $\pi_1(N)$ such that $N = \mathbb{H}^2/\rho_N$ then $\rho_M(\gamma_i^+)$ is hyperbolic (i.e. has two fixed points) whilst $\rho_N \circ f_*(\gamma_i^+)$ is parabolic (i.e. has a single fixed point.) Conversely if $\rho_N \circ f_*(\alpha), \alpha \in \pi_1$ is parabolic then α is conjugate to a power of γ_i^+ for some i.

After possibly perturbing f in its homotopy class we may assume that it is a K-Lipschitz diffeomorphism for some $K > 0$ so that, in particular, the boundary map, $\Psi : \partial\mathbb{H}^2 \to \partial\mathbb{H}^2$, of a lift $\hat{f} : \mathbb{H}^2 \to \mathbb{H}^2$, is well defined. Since we have adopted the philosophy of working with actions of groups on S^1 we want to understand what the operation on S^1 induced by pinching is; that is what is the boundary Ψ like? Recall that if $a, b, c, d \in S^1$ are four distinct points we say that $\{a, b\}$ *separates* $\{c, d\}$ iff c, d are in different connected components of $S^1 \setminus \{a, b\}$. A conjugation Φ preserves separation since it is a bijection and so preserves or reverses the cyclic order on S^1. A semiconjugation Ψ induced by an orientation preserving homeomorphism $\hat{f}$ preserves separation whenever the four points $\Psi(a), \Psi(b), \Psi(c), \Psi(d)$ are distinct. In this case we say that Ψ *weakly preserves separation.* As a corollary Ψ weakly preserves the cyclic ordering on the circle.

Let Λ_M, respectively Λ_N, denote the limit set of ρ_M, respectively ρ_N. The regular set of ρ_M consists of countably many intervals whilst $\Lambda_N = \partial\mathbb{H}^2$ since N is finite area. If I is an connected component of the regular set of ρ_M then there is $\alpha \in \pi_1$ conjugate to a γ_i^+ such that the endpoints are the fixed points of $\rho_M(\alpha)$. The Ψ sends the fixed points of $\rho_M(\alpha)$ to the fixed point of $\rho_N(f_*(\alpha))$ so that, since Ψ weakly preserves the order, $\Psi(I)$ is collapsed to a point too.

Lemma 8. ***Shape of limit sets*** *Let Λ_M, respectively Λ_N , denote the limit set of ρ_M, respectively ρ_N, and let Ψ be the map constructed above.*

(1) *Λ_N is the whole circle $\partial\mathbb{H}^2$.*
(2) *$\Psi(x)$ is a parabolic point iff x is in the regular set or is a peripheral point in Λ_M. The image of the peripheral points Λ^{per} for ρ_M is the set of peripheral points for ρ_N*
(3) *$\Psi^{-1}(\{y\})$ consists of more than one point iff y is parabolic. Moreover, $\Psi^{-1}(\{y\}) \cap \Lambda_M$ consists of exactly two peripheral points if y is parabolic and a single point otherwise.*

Proof. Evident from the preceding discussion □

11. Proof of Lemma 4

We just need to see now how the set $\hat{X}(p)$ behaves under our semi conjugacy Ψ to have all the elements of the proof of Theorem 3. The following is essentially well known but we include it for completeness:

Lemma 9. ***Simples preserved*** *Let Ψ be the quasi conjugacy between ρ_M and ρ_N constructed in the previous section. If $p \in \Lambda_M$ a peripheral point then*

(1) $\Psi(\hat{X}(p)) = \hat{X}(\Psi(p))$
(2) $\Psi(\hat{X}_{ess}(p)) = \hat{X}_{ess}(\Psi(p))$
(3) *Every peripheral point of $lX(p)$ is isolated so that $lX_{ess}(p)$ is closed.*

Proof. For (1) we first show that $\Psi(\hat{X}(p)) \subset \hat{X}(\Psi(p))$. Recall that, by definition, $x \in \hat{X}(p)$ iff $\{p, x\}$ projects to a geodesic α with no self intersections. Suppose that $\Psi(x) \notin \hat{X}(\Psi(p))$ that is $\{\Psi(p), \Psi(x)\}$ projects to a geodesic with self intersection on

$\mathbb{H}^2/\rho_N$. This means there is an element $h \in \pi_1$ such that $\rho_N(h)(\Psi(p)), \rho_N(h)(\Psi(x))$ separates $\Psi(p), \Psi(x)$, i.e. $\Psi(\rho_N(h)(p)), \Psi(\rho_N(h)(x))$ separates $\Psi(p), \Psi(x)$. Since Ψ preserves separation $\rho_M(h)(p), \rho_M(h)(x)$ separate p, x and the intersection of $\{\rho_M(h)(p), \rho_M(h)(x)\}$ and $\{p, x\}$ gives a double point for α on M that is $x \notin \hat{X}(p)$

To see that $\Psi |_{\hat{X}(p)}$ is a surjection observe that for every y, $\Psi^{-1}(\{y\})$ contains a point $x \in \Lambda_M$ and the same argument as above shows that $x \in X(p)$.

Part (2) follows from Lemma 8 (2). For $X_{ess}(p)^c$ is contained in the peripheral points of Λ_M so $\Psi(\hat{X}(p)^c)$ is contained in the parabolic (= peripheral) points of Λ_N, where the superscript c denotes the complement. So that

$$\begin{aligned} \Psi(X_{ess}(p)^c) \cap \Psi(\hat{X}_{ess}(p)) &= \emptyset \quad * \\ \Psi(X_{ess}(p)^c) \cap \hat{X}_{ess}(\Psi(p)) &= \emptyset \quad ** \end{aligned}$$

Now by part (1) and (*),

$$\hat{X}_{ess}(\Psi(x)) \sqcup \hat{X}_{ess}(\Psi(x))^c = \hat{X}(\Psi(p)) = \Psi(X(p)) = \Psi(\hat{X}_{ess}(p)) \sqcup \Psi(\hat{X}_{ess}(p)^c),$$

and (**) implies $\Psi(\hat{X}_{ess}(p)) = \hat{X}_{ess}(\Psi(p))$.

Finally, suppose $x \in \hat{X}(p)$ is peripheral and that there exists $x_n \in \hat{X}(p)$ converging to x. Now $\Psi(x_n)$ converges to a parabolic point $\Psi(x) \in \hat{X}(\Psi(p))$. By [5] $\Psi(x)$ is isolated and so $\Psi(x_n)$ is stationnary, that is for all n sufficiently large $\Psi(x_n) = \Psi(x)$. But $\Psi^{-1}(\{\Psi(x_n)\}) \cap \hat{X}(p) = \Psi^{-1}(\{\Psi(x)\}) \cap X(p)$ consists of at most two points (Lemma 8(3)) one of which is x. Thus x_n is stationnary too and x is isolated. □

Proof of Lemma 4: Suppose there is a gap I in $\hat{X}_{ess}(p)$ and let $\{a, b\}$ be the frontier of I. Now since $\hat{X}_{ess}(p)$ is closed $a, b \in \hat{X}_{ess}(p)$ and $\Psi(a), \Psi(b) \in \hat{X}_{ess}(\Psi(p))$ by Lemma 9 (2). Observe that a is not a peripheral point so $\Psi(a)$ is not a parabolic point so $\Psi(a) \neq \Psi(b)$ and $\Psi(I)$ has non empty interior. Since Ψ preserves the cyclic order on $\partial\mathbb{H}^2$ there are no points of $\Psi(\hat{X}_{ess}(p))$ in $\Psi(I)$ and since $\hat{X}(\Psi(p)) = \Psi(\hat{X}_{ess}(p))$ this is a gap in $\hat{X}(\Psi(p))$. By Theorem 3 (3) $\Psi(I)$ contains a parabolic point $p' \in \hat{X}(\Psi(p))$ and $\Psi^{-1}(\{p'\})$ contains a pair of peripheral points by Lemma 8 (3). At least one of these is in $\hat{X}(p)$ since otherwise, using the same argument as in Lemma 9 (1) the geodesic joining $\Psi(p), p'$ projects to a geodesic with a double point on N. This completes the proof. □

12. Closing Philosophical Remarks

Informally, our point of view is that Mirzakhani's identity is to the original identity 1 as Taylor's series are to MacLaurin's series. A priori Taylor's series are a generalization of MacLaurin's series but in practice one deduces Taylor's theorem by change of coordinates $x \mapsto x + a$. The role of this change of coordinates is taken by the choice of a semi conjugacy and Theorem 3 is a corollary of Theorem 4 of [5] (see Section 7,8). On the other hand to explain the background geometry behind the trigonometric identities takes more time.

The original identity was a "happy accident" in the author's attempts to understand Thurston's *stretch maps construction* [10] and Penner's *arc complex* [11] and most of the exposition in Section 5 dates from that time. The theory has subsequently been elaborated and improved by Bonahon and his students [3][13] see also [4] for background on laminations. The shear space of an ideal triangulation is probably the natural place to understand Mirzakhani's work on volumes for the following reason. By work of Penner [12] the Weil-Petterson volume form on the

Teichmueller space of a surface of genus $g > 1$ with $n > 0$ cusps is the pull-back via the shear coordinate map of (a constant multiple of) the usual volume form on $\mathbb{R}^{6g-6+2n}$.

The approach to proving and generalizing identities via the limit set of (quasi fuchsian) groups has been developped by Sakuma et al following work of Bowditch . We do not need to be as careful with our semi conjugacies as they are since we know that for all our surfaces the Birman-Series theorem holds, so that $X(p)$ is Hausdorff dimension 0 *a priori*. In general, one must check that the the semi conjugacy is Holder continuous as this implies that the image of a set of Hausdorff dimension 0 is again Hausdorff dimension 0. This is essentially what is used to prove the identities in the quasi-fuchsian case where length is replaced by complex length. Our contribution here is to adapt the definitions so that Theorem 4 of [5] "becomes" Theorem 2 above.

One can prove Theorem (2) under the hypothesis that there is a semi conjugacy Ψ with a fuchsian group such that the quotient surface has only cusps, and such that Ψ satisfies the hypothesis of Lemma 8. It is relatively easy to show that such a Ψ exists for the action at infinity of the fundamental group of a surface of finite type with non empty boundary and a metric of pinched variable curvature; here infinity is the ideal boundary of the metric universal cover. However, it is clear that the gap functions are only interesting in the constant curvature case.

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