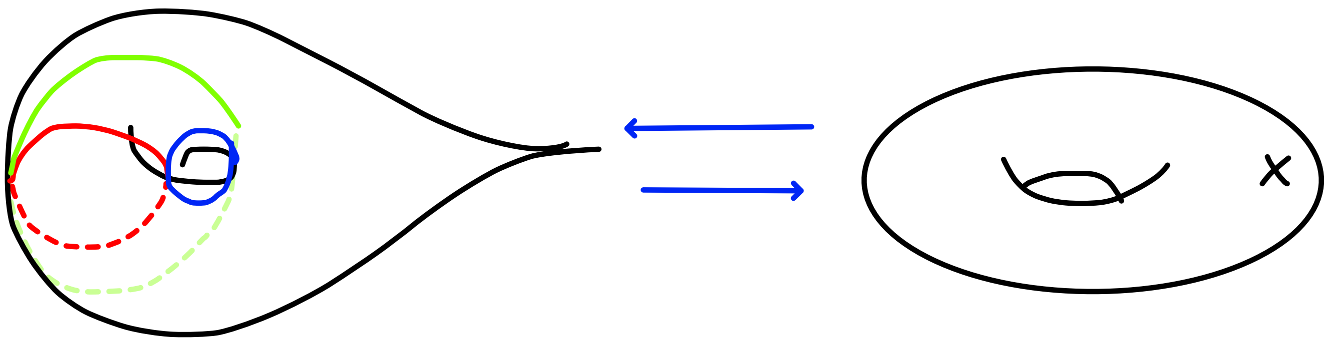
Reading Wolpert's article

The Teichmüller space

We begin with an exposition of the Teichmüller theory of the once punctured torus. Our goal is to describe coordinates for the Teichmüller space, and to describe the action of the modular group. The material was in part previously considered by Keen, [3].

A once punctured torus is uniformized by a Fuchsian group Γ, $\Gamma \subset PSL(2;\mathbb{R})$. We shall use the following normalized form for the presentation of Γ. Hyperbolic transformations $A, B \in PSL(2;\mathbb{R})$ freely generate Γ with $ABA^{-1}B^{-1}$ parabolic; the repelling (resp. attracting) fixed point of A is 0 (resp. ∞) and the attracting fixed point of B is 1. In fact the group Γ can be lifted into $SL(2;\mathbb{R})$ such that tr A, tr B, tr AB become positive, where tr denotes the trace of a matrix. We shall consider Γ both as a subgroup of $SL(2;\mathbb{R})$ and of $PSL(2;\mathbb{R})$ without making the proper distinction. The quantities $x = \operatorname{tr} A$, $y = \operatorname{tr} B$, and $z = \operatorname{tr} AB$ uniquely characterize the above description of Γ. The transformation $ABA^{-1}B^{-1}$ is parabolic. An elementary argument shows that the commutator $ABA^{-1}B^{-1}$ has negative trace and consequently $\operatorname{tr} ABA^{-1}B^{-1} = -2$. The equation $\operatorname{tr} ABA^{-1}B^{-1} = -2$ is equivalent to the identity $x^2 + y^2 + z^2 = xyz$. This is the unique relation satisfied by the triple (x, y, z).

THEOREM (Fricke Klein [2], Keen [3]). *The Teichmüller space $\mathcal{T}$ of the once punctured torus is the sublocus of $x^2 + y^2 + z^2 = xyz$ satisfying $x, y, z > 2$.*



the triple $A, B, C \longleftrightarrow$ triple simple closed curves

$\longleftrightarrow \alpha, \beta, \alpha\beta \in \pi_1$

Computations of the Aut$(\mathbb{Z} * \mathbb{Z})$ action

non. It is the direct consequence of the non-uniqueness of a choice of generators in the presentation for Γ. The automorphism group of Γ will be used to study the different choices of generators for Γ. Let G be the free group with generators A and B. The automorphism group Aut (G) of G has generators, σ, P and U where

$$\begin{matrix} \sigma(A) = A^{-1} \\ \sigma(B) = B \end{matrix} \qquad \begin{matrix} P(A) = B \\ P(B) = A \end{matrix} \quad \text{and} \quad \begin{matrix} U(A) = AB \\ U(B) = B \end{matrix},$$

[4]. A representation of Aut (G) in $GL(2;\mathbb{Z})$ is obtained by letting Aut (G) act on $G/[G, G] \approx \mathbb{Z} \oplus \mathbb{Z}$. Choosing the cosets of A and B as generators for $G/[G, G]$ we have under the representation

$$\sigma \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad P \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad U \to \begin{Bmatrix} 1 & 1 \\ 0 & 1 \end{Bmatrix}.$$

We shall be concerned with $\mathrm{Aut}^+(G)$ the preimage of $SL(2;\mathbb{Z}) \subset GL(2;\mathbb{Z})$. Denote by Inn (G) the inner automorphism group of G. The essential properties of the representation are given in the following theorem of Nielsen, [4].

THEOREM. *Let G be the free group on two generators. Then*

$$\mathrm{Out}^+(G) = \mathrm{Aut}^+(G)/\mathrm{Inn}(G) \approx SL(2;\mathbb{Z}).$$

Ex Calculate the actions of the elements of Out$(\mathbb{Z} * \mathbb{Z})$ on the character variety

$$G \quad \begin{matrix} A \\ B \\ AB \end{matrix} \mapsto \begin{matrix} A^{-1} \\ B \\ A^{-1}B \end{matrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{matrix} \mathrm{tr}\, A \\ \mathrm{tr}\, B \\ \mathrm{tr}\, AB \end{matrix} \mapsto \begin{matrix} \mathrm{tr}\, A^{-1} \\ \mathrm{tr}\, B \\ \mathrm{tr}\, A^{-1}B \end{matrix} = \begin{pmatrix} x \\ y \\ xy - z \end{pmatrix}$$

We now wish to focus our attention on the principal congruence subgroup $\Gamma(2)$ of level 2 in $SL(2;\mathbb{Z})$, where

$$\Gamma(2)=\left\{C\in SL(2;\mathbb{Z})\mid C\equiv\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\bmod 2\right\}.$$

Denote by $P\Gamma(2)$ the image of $\Gamma(2)$ in $PSL(2;\mathbb{Z})$. By an elementary argument the indices satisfy

$$[SL(2;\mathbb{Z}):\Gamma(2)]=[PSL(2;\mathbb{Z}):P\Gamma(2)]=6.$$

We wish to characterize the preimage of $P\Gamma(2)$ in $\mathrm{Aut}(G)$. Let $\mathcal{M}_2$ be the subgroup of $\mathrm{Aut}(G)$ generated by $\rho_1=\sigma U^2$, $\rho_2=\sigma$ and $\rho_3=P\sigma U^2P$. Under the representation of $\mathrm{Aut}(G)$ in $GL(2;\mathbb{Z})$

$$\rho_1\to\begin{pmatrix}-1 & -2\\ 0 & 1\end{pmatrix}\qquad \rho_2\to\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\quad\text{and}\quad \rho_3\to\begin{pmatrix}1 & 0\\ -2 & -1\end{pmatrix}.$$

Ex What is the index of $\Gamma(2)$ in $SL(2,\mathbb{Z})$

Denote by $\mathcal{M}_2^+$ the intersection $\mathcal{M}_2 \cap \mathrm{Aut}^+(G)$. Now the representations of ρ_1, ρ_2 and ρ_3 each have negative determinant; consequently $\mathcal{M}_2^+ \subset \mathcal{M}_2$ is the subgroup of words in ρ_1, ρ_2 and ρ_3 of even length. Under the representation of $\mathrm{Aut}(G)$ we claim that $\mathcal{M}_2^+ \subset \Gamma(2)$, and that $\mathcal{M}_2^+$ surjects onto $P\Gamma(2)$. Indeed the representations of ρ_1, ρ_2 and ρ_3 are each congruent to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ mod 2, the inclusion $\mathcal{M}_2^+ \subset \Gamma(2)$ is immediate. The images of $\rho_2\rho_1$ and $\rho_2\rho_3$ are respectively $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}$ the lifts of the generators of $P\Gamma(2)$. We shall establish below that $\mathcal{M}_2^+$ acts effectively and thus conclude that the natural map $\mathcal{M}_2^+ \to P\Gamma(2)$ is a bijection.

First we shall consider the action of $\mathcal{M}_2$ in the (x, y, z) coordinates. We begin with the action of ρ_1, ρ_2 and ρ_3 on the generators A, B of Γ

$$\rho_1(A) = B^{-2}A^{-1} \qquad \rho_2(A) = A^{-1} \qquad \rho_3(A) = A$$
$$\rho_1(B) = B \qquad \rho_2(B) = B \qquad \rho_3(B) = A^{-2}B^{-1}$$

There is natural induced action on the traces of the generators of Γ. Recalling that $x = \mathrm{tr}\, A$, $y = \mathrm{tr}\, B$, $z = \mathrm{tr}\, AB$ we have

$$\rho_1(x) = yz - x \qquad \rho_2(x) = x \qquad \rho_3(x) = x$$
$$\rho_1(y) = y \qquad \rho_2(y) = y \qquad \rho_3(y) = xz - y$$
$$\rho_1(z) = z \qquad \rho_2(z) = xy - z \qquad \rho_3(z) = z$$

Ex Find the elements of $\mathrm{Out}(\mathbb{Z} * \mathbb{Z})$ that induce these automorphisms