Action of McG on simple geodesics

$(\Sigma, *)$ topo surface $* =$ base pt

$(X, *)$ hyp surface

$\pi \ (\hat{X}, \tilde{x}) \to (X, x)$ cover

$\varphi \quad X \to X$ homeo $\varphi(x) = x$

$\exists! \quad \tilde{\varphi} \quad \hat{X} \to \hat{X}$ homeo $\tilde{\varphi}(\tilde{x}) = \tilde{x}$

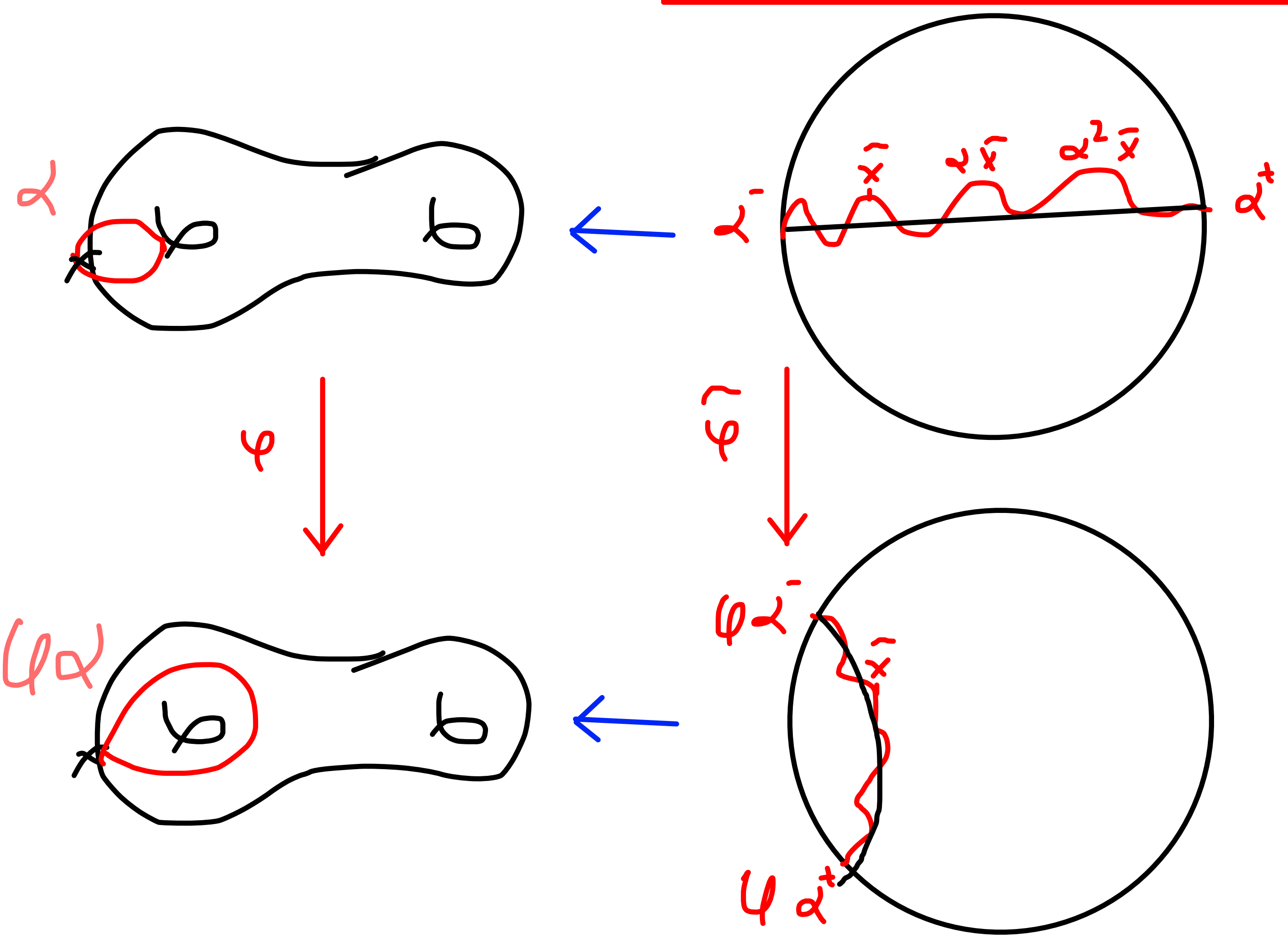
Curve straightening

$$\begin{array}{ccccc} \tilde{x} \in \tilde{\alpha} & \subset & \tilde{X} = \mathbb{H}^2 \\ \downarrow & & \downarrow \\ x \in \alpha & \subset & X \end{array}$$

MORSE LEMMA

$\hat{\alpha} \ \mathbb{R} \to \mathbb{H}^2$ Lipschitz proper

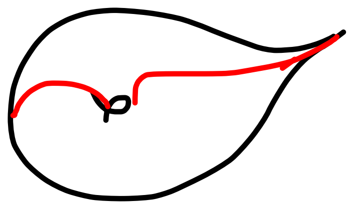
$\exists! \ [\hat{\alpha}^-, \hat{\alpha}^+]$ geodesic at bdd distance from $\hat{\alpha}$



action on 2 types of curves (geodesics) in punctured torus $= \mathbb{T}^\times$

1/ simple closed

2/ simple bicuspidal

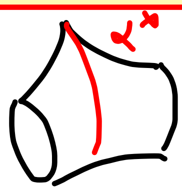


Prop $0 \neq [\alpha] \in H_1(\mathbb{T}^\times, \mathbb{Z}) \simeq \mathbb{Z}^2$ primitive
then $\exists^1$ simple geodesic representing $[\alpha]$

proof lift to homology cover viewed as $\mathbb{R}^2 \setminus \mathbb{Z}^2$

Prop α closed simple $\subset \mathbb{T}^\times$
$\exists^1$ α^* bicuspidal disjoint from α

<u>proof</u> cut along α $\to$



= annulus with marked pt

Ex. Let J $\mathbb{T}^\times \circlearrowleft$ be the elliptic involution
we know J_* $H_1(\mathbb{T}^\times, \mathbb{Z}) \circlearrowleft$ is $-I_2$

show that 1) J maps each α simple closed to itself

2) deduce that α passes through 2 of the 3 fixed pts of J

3) what about α^* ?

Shimuza's Lemma and cusp regions

Lemma Let $\Gamma < SL(2,\mathbb{R})$ discrete

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma \Rightarrow \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $c = 0$ or $|c| \geq 1$

Def X hyp surface non compact

cusp region $\subset X$ isometric to $\{ z \in \mathbb{H}^2, \operatorname{Im} z \geq h \} / \langle z \mapsto z+1 \rangle$

Exo area = perimeter = h

horocyclic foliation = partition of cusp region into horocycles

$$F_t = \{ z \in \mathbb{H}^2, \operatorname{Im} z = t \geq h \} / \langle z \mapsto z+1 \rangle$$

Lemma Let $\mathbb{H}^2/\Gamma = \mathbb{T}^\times$

i) $\exists$ a cusp region H of area 2

ii) α simple closed $\Rightarrow \alpha \cap H = \emptyset$

iii) $\alpha^\times$ simple bicuspidal

$\Rightarrow \alpha^\times$ meets horocyclic foliation perpendicular

The set of all simple bicuspidal geodesics

Let H be as in the lemma

$$\overline{F_t \cap \bigcup_{\alpha^\times} \alpha^\times} = \underbrace{\text{Kantor set}}_{K} \cup \text{isolated pts}$$

Fact 1) x isolated $\Leftrightarrow \exists\ \alpha^\times,\ x \in \alpha^\times$

2) K^c consists of countably many intervals each of which contains exactly one isolated pt