3. Automorphisms of $\mathbb{Z} * \mathbb{Z}$

Let $\mathbb{Z} * \mathbb{Z}$ denote the free group on 2 generators which we will denote α and β. We think of α and β as simple loops meeting in a single point in a holed torus. An element $\gamma \in \mathbb{Z} * \mathbb{Z}$ is called *primitive*, if there exists an automorphism ϕ, such that $\gamma = \phi(\alpha)$. If $\phi(\delta) = \beta$, then γ and δ are called *associated primitives*.

Following [11] let $\phi : \mathbb{Z} * \mathbb{Z} \to \mathbb{Z}^2$ be the canonical abelianizing homomorphism. The kernel of ϕ is a *characteristic subgroup* that is it is $\mathrm{Aut}(\mathbb{Z}*\mathbb{Z})$-invariant so there is a homorphism from $\mathrm{Aut}(\mathbb{Z}*\mathbb{Z})$ to the automorphisms of $\mathbb{Z}^2$ namely $\mathrm{GL}(2, \mathbb{Z})$. In fact this homomorphism is surjective and the kernel is exactly the inner automorphisms so that it induces an isomorphism between the group of outer automorphisms of $\mathbb{Z} * \mathbb{Z}$ and $\mathrm{GL}(2, \mathbb{Z})$.

(1) If γ is primitive then $\phi(\gamma)$ is a primitive element of $\mathbb{Z}^2$.
(2) $\mathrm{Aut}(\mathbb{Z} * \mathbb{Z})$ acts transitively on primitive elements of $\mathbb{Z} * \mathbb{Z}$ and $\mathrm{GL}(2, \mathbb{Z})$ acts transitively on primitive elements of $\mathbb{Z}^2$.

3.1. $\mathbb{Z} * \mathbb{Z}$ as a surface group.

The group $\mathbb{Z} * \mathbb{Z}$ is isomorphic to the fundamental group of exactly 4 surfaces, namely

- the three holed sphere
- the one holed mobius band (the two holed projective plane)
- the one holed klein bottle

3.2. **Topological realizations of automorphisms of $\mathbb{Z} * \mathbb{Z}$.** It is often useful to identify $\mathbb{Z} * \mathbb{Z}$ with the fundamental group of the holed torus and think of $\mathrm{Out}(\mathbb{Z} * \mathbb{Z})$, the outer automorphism group of $\mathbb{Z} * \mathbb{Z}$, as being the mapping class group of the holed torus. The outer automorphism group is generated by the so-called *Nielsen transformations*, which either permute the basis α, β, or transform it into $\alpha\beta, \beta$, or $\alpha\beta^{-1}, \beta$. Both these latter transformations can be realized topologically as Dehn twists on the holed torus.

In what follows we will identify $\mathbb{Z}^2$ with the homology of the surface Σ and will call the cover $\tilde{\Sigma} \to \Sigma$ corresponding to $\ker \phi$ the *homology cover or maximal abelian of* Σ.

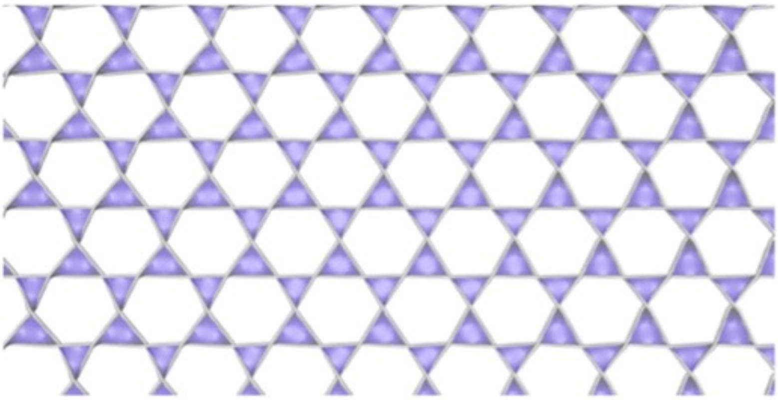


FIGURE 9. Maximal abelian cover for the three holed sphere. The boundary loops lift to the grey curves, the patches appear triangular but in fact are hexagons joined along sides which are perpendicular to the plane.

There are many fine expositions of the action of $\mathrm{Out}(\mathbb{Z} * \mathbb{Z})$ on the maximal abelian cover of the one holed torus. Unfortunately the action

Sketch of how to do it

The punctured plane $\mathbb{R}^2 \setminus \mathbb{Z}^2$ can be identified with the maximal abelian cover of the punctured torus: if $\Gamma \simeq \mathbb{Z}^2$ denotes the group of deck transformations then the punctured torus is obtained as a quotient:

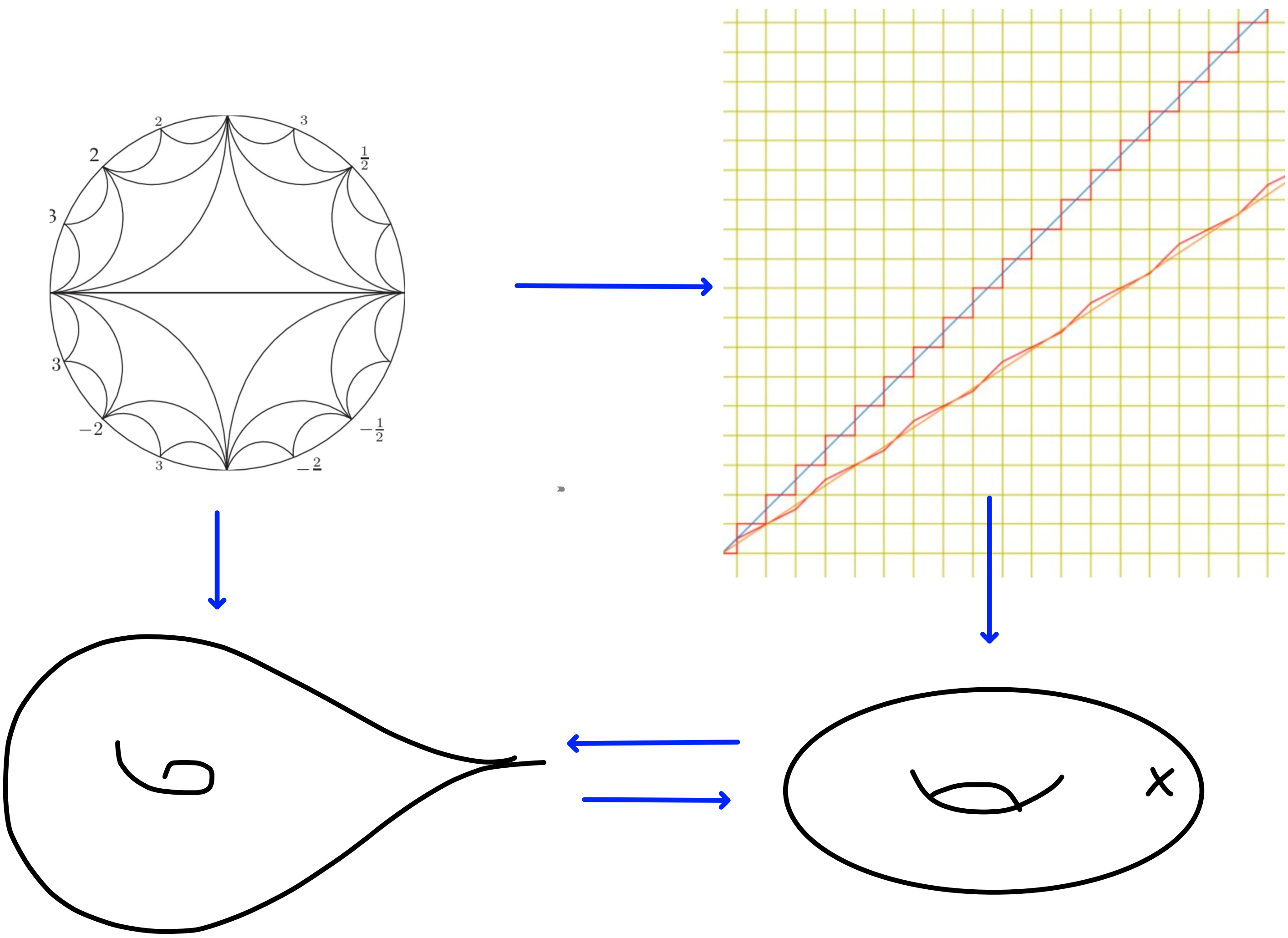
$$\left(\mathbb{R}^2 \setminus \mathbb{Z}^2\right)/\Gamma.$$

There is a Γ-equivariant retraction H_t from the punctured plane to a grid of horizontal and vertical lines (see Figure (12)). So this grid is in fact the set of lifts of a spine for the once punctured torus.

Recall that a *Weierstrass point* on the punctured torus is a fixed point of the *elliptic involution* that is the involution of $\left(\mathbb{R}^2 \setminus \mathbb{Z}^2\right)/\Gamma$ induced by $\mathbb{R}^2 \to \mathbb{R}^2$, $v \mapsto -v$. It is easy to see that the set of lifts of these points are respectively $\mathbb{Z}^2 + (\frac{1}{2}, 0)$, $\mathbb{Z}^2 + (0, \frac{1}{2})$ and $\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$. So the grid contains lifts of the three Weierstrass points - these occur at the vertices of the grid and the midpoints of the horizontal and vertical edges. We may also identify the grid with the lift of a spine for the pants. We quite simply choose a figure eight curve and a retraction

Universal cover

Homology cover



Action on the character variety

1.1. **Hyperbolic surfaces.** Each of these four surfaces admits a (non unique) hyperbolic structure, in fact there is a three dimensional deformation space of such structures. Indeed it has been known since the time of Fricke that, as a consequence of Rieman's Uniformization Theorem, the three holed sphere and the one holed torus the moduli space of structures can be identified with a semi algebraic subset of $\mathbb{R}^3$. More precisely, one obtains a hyperbolic structure X on Σ from a discrete faithful representation ρ of $\pi_1(\Sigma)$ into $\mathrm{isom}(\mathbb{H})$ such that $\mathbb{H}/\rho(\mathbb{Z}*\mathbb{Z})$ is homeomorphic to Σ. If Σ is orientable one can lift ρ from a representation into $\mathrm{isom}(\mathbb{H})^+$, which is isomorphic to $\mathrm{PSL}(2,\mathbb{Z})$ to a representation, $\tilde{\rho} : \mathbb{Z}*\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$. Doing this allows one to consider the trace $\mathrm{tr}\,\tilde{\rho}(\gamma) \in \mathbb{R}$ for any element γ of the free group so that many problems can be reduced to questions of linear algebra. In particular one defines a character map

$$\chi : X \mapsto (\mathrm{tr}\,\tilde{\rho}(\alpha), \mathrm{tr}\,\tilde{\rho}(\beta), \mathrm{tr}\,\tilde{\rho}(\alpha\beta)),$$

which provides an embedding of the moduli space of hyperbolic structures on Σ. Goldman [9] studied the dynamical system defined by the action of the group of outer automorphisms of $\mathbb{Z}*\mathbb{Z}$ on the moduli space. If ϕ is an automorphism then it acts on the representations

$$\tilde{\rho} \mapsto \tilde{\rho} \circ \phi^{-1}$$

and evidently this induces an action on the points in the image of the embedding above. It turns out that, by applying Cayley-Hamilton theorem to $\mathrm{SL}(2,\mathbb{Z})$, it is easy to show that this action is the restriction of polynomial diffeomorphisms of $\mathbb{R}^3$. Later Goldman and his collaborators [8] made a similar study for the two-holed projective plane and the one-holed Klein bottle. This latter work is quite delicate for two reasons. Firstly, the representation ρ no longer lifts to $\mathrm{SL}(2,\mathbb{R})$ but $\mathrm{SL}(2,\mathbb{C})$ and secondly the automorphisms will in general no longer be geometric for these surfaces.

$$x^2 + y^2 + z^2 - xyz = 0$$

Automorphisms of (Markoff) cubic

$$x^2 + y^2 + z^2 - 3xyz = 0$$

1.2.1. *Reduction theory.* The fact that the Markoff triples form a single orbit is a corollary of a classical result of Markoff in reduction theory which we now explain briefly. We start by giving the Markoff triples an ordering, in the obvious way, using the sup norm on $\mathbb{R}^3$ and proceed to show that any solution can be obtained from $(1,1,1)$ by repeatedly applying generators of the group of automorphisms of the cubic. This automorphism group can be shown to be generated by:

(1) sign change automorphisms $(x, y, z) \mapsto (x, -y, -z)$.
(2) coordinate permutations eg $(x, y, z) \mapsto (y, x, z)$.
(3) a Vieta flip $(x, y, z) \mapsto (x, y, 3xy - z)$.

Since Markoff triples are solutions in positive integers we will use just the *Markoff morphisms* that is the group generated by permutations and the Vieta flip. Given a Markoff triple $(x, y, z) \neq (1, 1, 1)$ we may apply a permutation so that $x \leq y < z$ which one can try to "reduce" using the Vieta flip, that is replacing the it by the triple $(x, y, 3xy - z)$. We obtain a smaller solution provided $3xy < 2z$ and this inequality holds for every Markoff triple *except* $(1, 1, 1)$. The reduction process gives rise to the structure of a rooted binary tree, the *Markoff tree*, on the Markoff triples with $(1, 1, 1)$ as the *root triple*.