10

Spaces of discrete groups

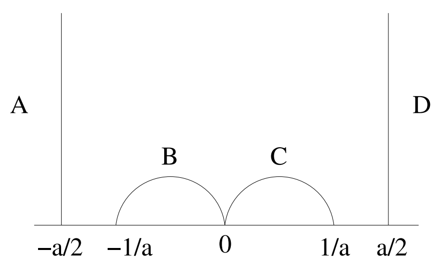
10.1 Non rigidity phenomena for subgroups of PSL$(2,\mathbb{R})$

We consider first the family of Hecke groups which we already encountered before,

$$\Gamma_a = \left\langle \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \right\rangle \subset \mathrm{PSL}(2,\mathbb{R}).$$

One knows that all Γ_a are free if $|a| \geq 2$ hence isomorphic to each other. If $a \in \mathbb{Z}$ then $\Gamma_a \subset \mathrm{PSL}(2,\mathbb{Z})$ and thus they are also discrete. However, we claim that Γ_2 and Γ_a, $a > 2$ are not conjugate inside PSL$(2,\mathbb{R})$.

Using the Poincaré theorem we can construct Γ_a by making use of a fundamental regions. It is easy to verify that the domain P_a in the figure below is a fundamental region for Γ_a having the sides A, B, C, D.



Let $\iota : \{A, B, C, D\} \to \{A, B, C, D\}$ be the involution given by $\iota(A) = D$, $\iota(B) = C$, and let us define $\sigma_A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $\sigma_B = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$. Then Γ_a is naturally identified with the group generated by σ_A and σ_B.

Now, if $a > 2$ then $\mu(P_a) = +\infty$, while for $a = 2$ we have $\frac{a}{2} = \frac{1}{a}$ and thus the quotient surface is a non-compact cusped surface of finite volume. Thus, $\mathrm{vol}(\mathcal{H}/\Gamma_2) \neq \mathrm{vol}(\mathcal{H}/\Gamma_a)$ and thus Γ_2 and Γ_a cannot be conjugate.

Let now consider the group $\Gamma_2' = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle \subset \mathrm{PSL}(2,\mathbb{R})$. Then Γ_2' has the same fundamental region P_2 as Γ_2. Moreover the involution ι :

$\{A, B, C, D\} \to \{A, B, C, D\}$ which yields Γ_2' is different from the previous one, namely $\iota(A) = C$, $\iota(B) = D$. If we consider the matrices ${\sigma'}_A^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, $\sigma_B' = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ then one verifies easily that Γ_2' is the group generated by σ_A' and σ_B'. The Poincaré theorem implies that Γ_2' is also free.

However, despite the fact that Γ_1 and Γ_2 share the same fundamental region and thus $\mu(\mathcal{H}/\Gamma_2) = \mu(\mathcal{H}/\Gamma_2')$, these groups are not conjugate within $\mathrm{PSL}(2, \mathbb{R})$. The reason is that $\mathcal{H}/\Gamma_2$ is homeomorphic to a 2-sphere with 3 cusps (i.e. $S^2 - \{0, 1, \infty\}$) while $\mathcal{H}/\Gamma_2'$ is a torus with 1-cusp. This follows immediately by looking at the identifications of sides of the respective fundamental domains induced by the involution.

If one seeks for families of Fuchsian groups then one needs to fix both the isomorphism type of the abstract group Γ as well as the homeomorphism type of the quotient surface $\mathcal{H}/\Gamma$. If Γ has no elliptic points then $\mathcal{H}/\Gamma$ is an orientable surface, with the orientation inherited by taking the quotient.

Definition 27. *The* Teichmüller space $\mathcal{T}(\Sigma)$ *of the oriented surface Σ is the space of marked Fuchsian groups $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ such that $\pi_1\Sigma \to \Gamma$ is an isomorphism and Σ is orientation-preserving homeomorphic to $\mathcal{H}/\Gamma$. Notice that a marking of Γ is provided by a system of generators.*

An equivalent definition is to set:

Definition 28. *The Teichmüller space $\mathcal{T}(\Sigma)$ is the set of marked complex structures on Σ up to the equivalence relation below. A marked complex structure is a homotopy equivalence $f : \Sigma \to M$ where M is an arbitrary Riemann surface and two such f and $f' : \Sigma \to M'$ are equivalent $f \sim f'$ if there exist a conformal equivalence $h : M \to M'$ such that $f' \simeq f \circ h$, $\simeq$ denoting homotopy equivalence.*

By Riemann's uniformization theorem we can always write $M = \mathcal{H}/\Gamma$ where $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ acts by isometries. In particular, we have an identification:

$$\mathcal{T}(\Sigma) = \mathrm{Hom}^+_{f,d}(\pi_1, \Sigma, \mathrm{PSL}(2, \mathbb{R}))/\text{conjugacy within } \mathrm{PSL}(2, \mathbb{R})$$

where $\mathrm{Hom}^+_{f,d}(\pi_1, \Sigma, \mathrm{PSL}(2, \mathbb{R}))$ denotes the space of *faithful* homomorphisms $\varphi : \pi_1\Sigma \to \mathrm{PSL}(2, \mathbb{R})$ such that $\varphi(\pi_1(\Sigma)) = \Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ is a Fuchsian group and if φ preserves the orientation i.e. the induced homeomorphism $\Sigma \to \mathcal{H}/\Gamma$ is preserving the orientation.

10.2 Thurston-Bonahon-Penner-Fock coordinates on the Teichmüller spaces

10.2.1 Preliminaries on fatgraphs

Let Γ be a finite graph. We denote by V_Γ and E_Γ the set of its vertices and edges respectively.

Definition 29. *An* orientation *at a vertex v is a cyclic ordering of the (half-) edges incident at v. A* fatgraph *(sometimes called* ribbon graph*) is a graph endowed with an orientation at each vertex of Γ. A* left-hand-turn path *in Γ is a directed closed path in Γ such that if e_1, e_2 are successive edges in the path meeting at v, then e_2, e_1 are successive edges with respect to the orientation at v. The ordered pair e_1, e_2 is called a left-turn. We sometimes call* faces *of Γ the left-hand-turn paths and denote them by F_Γ.*

A fatgraph is usually represented in the plane, by assuming that the orientation at each vertex is the counter-clockwise orientation induced by the plane, while the intersections of the edges at points other than the vertices are ignored. There is a natural surface, which we denote by Γ^t obtained by thickening the fatgraph. We usually call Γ^t the ribbon graph associated to Γ. We replace the half-edges around a vertex by thin strips joined at the vertex, whose boundary arcs have natural orientations. For each edge of the graph we connect the thin strips corresponding to the vertices by a ribbon which follows the orientation of their boundaries. We obtain an oriented surface with boundary. The boundary circles are in one-to-one correspondence with the left-hand-turn paths. If one caps each left-hand-turn path by a 2-disk we find a closed surface Γ^c, and this explains why we called these paths faces. The centers of the 2-disks will be called punctures of Γ^c and $\Gamma^o = int(\Gamma^t)$ is homeomorphic to the punctured surface.

There is a canonical embedding $\Gamma \subset \Gamma^t$, and one can associate to each edge e of Γ a properly embedded orthogonal arc $e^\perp$ which joins the two boundary components of the thin strip lying over e. The dual arcs $e^\perp$ divide the ribbon Γ^t into hexagons. When we consider the completion Γ^c, we join the boundary points of these dual arcs to the punctures within each 2-disk face and obtain a set of arcs connecting the punctures, denoted by the same symbols. Then the dual arcs divide Γ^c into triangles. We set $\Delta(\Gamma)$ for the triangulation obtained this way. The vertices of $\Delta(\Gamma)$ are the punctures of Γ^c. Remark that $\Delta(\Gamma)$ is well-defined up to isotopy. Now the fatgraph $\Gamma \subset \Gamma^t$ can be recovered from $\Delta(\Gamma)$ as follows. Mark a point in the interior of each triangle, and connect points corresponding to adjacent triangles. This procedure works for any given triangulation Δ of an oriented surface and produces a fatgraph $\Gamma = \Gamma(\Delta)$ with the property that $\Delta(\Gamma) = \Delta$. The orientation of Γ comes from the surface.

If Γ^o is the surface Σ_g^s of genus g with s punctures then by Euler characteristic reasons we have: $\sharp V_\Gamma = 4g - 4 + 2s$, $\sharp E_\Gamma = 6g - 6 + 3s$, $\sharp F_\Gamma = s$.

10.2.2 Coordinates on Teichmüller spaces

Marked ideal triangles

Let us recall that $\mathcal{D}$ denotes the unit disk, equipped with the hyperbolic metric. Recall that any two ideal triangles are isometric, since we may find a Möbius transformation, which takes one onto the other. Choose a point on each edge of the ideal triangle. The chosen points will be called *tick-marks*.

Definition 30. *A* marked ideal triangle *is an ideal triangle with a tick-mark on each one of its three sides. An isomorphism between two marked ideal triangles is an isomorphism between the ideal triangles which preserves the tick-marks.* A standard marked ideal triangle *is one which is isometric to the marked ideal triangle whose vertices in the disk model are given by* $v_1 = 1,\ v_2 = \omega,\ v_3 = \omega^2$ *and whose tick-marks are* $t_1 = -(2-\sqrt{3}),\ t_2 = -(2-\sqrt{3})\omega,\ t_3 = -(2-\sqrt{3})\omega^2$, *where* $\omega = \mathrm{e}^{2\pi \mathrm{i}/3}$.

The ideal triangle and its tick-marks are pictured in figure 10.1 in both the half-plane model and the disk model; they correspond each other by the map $z \mapsto \frac{z-(\omega+1)}{z-(\bar{\omega}+1)}$.

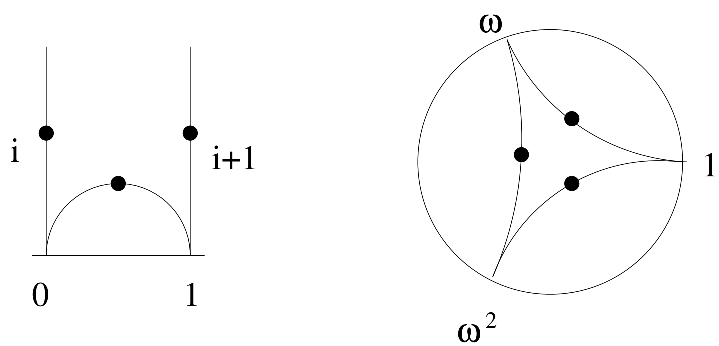


Fig. 10.1. The standard marked ideal triangle

Coordinates on the Teichmüller space of punctured surfaces

Set $\mathcal{T}_g^s$ for the Teichmüller space of the surface of genus g with s punctures. Let Γ be a fatgraph with the property that Γ^c is a surface of genus g with s punctures and let S denote the surface Γ^c endowed with a hyperbolic structure of finite volume, having the cusps at the punctures.

As already explained above we have a triangulation $\Delta(\Gamma)$ associated to Γ. One deforms the arcs of $\Delta(\Gamma)$ within their isotopy class in order to make them geodesic. We shall associate a real number $t_e \in \mathbb{R}$ to each edge of $\Delta(\Gamma)$ (equivalently, to each edge of Γ). Set Δ_v and Δ_w for the two triangles sharing the edge $e^{\perp}$. We consider next two adjacent lifts of these triangles (which we denote by the same symbols) to the hyperbolic space $\mathbb{H}^2$. Then both Δ_v and Δ_w are isometric to the standard ideal triangle of vertices v_1, v_2 and v_3. These two isometries define (by pull-back) canonical tick-marks t_v and respectively t_w

on the geodesic edge shared by Δ_v and Δ_w. Set t_e for the (real) length of the translation along this geodesic needed to shift t_v to t_w. Notice that this geodesic inherits an orientation as the boundary of the ideal triangle Δ_v in $\mathbb{H}^2$ which gives t_e a sign. If we change the role of v and w the number t_e is preserved.

An equivalent way to encode the translation parameters is to use the cross-ratios of the four vertices of the glued quadrilateral $\Delta_v \cup \Delta_w$, which are considered as points of $\mathbb{R}P^1$. It is convenient for us to consider $\mathbb{R}P^1$ as the boundary of the upper half-plane model of $\mathbb{H}^2$, and hence the ideal points have real (or infinite) coordinates. Let assume that Δ_v is the ideal triangle determined by $[p_0 p_{-1} p_\infty]$ and Δ_w is $[p_0 p_\infty p]$. We consider then the following cross-ratios:

$$z_e = [p_{-1}, p_\infty, p, p_0] = [p, p_0, p_{-1}, p_\infty] = \log - \frac{(p_0 - p)(p_{-1} - p_\infty)}{(p_\infty - p)(p_{-1} - p_0)}.$$

This cross-ratio reflects both the quadrilateral geometry and the decomposition into two triangles. In fact the other possible decomposition into two triangle of the same quadrilateral leads to the value z_e.

The relation between the two translation parameters t_e and z_e is immediate. Consider the ideal quadrilateral of vertices $-1, 0, e^z$ and ∞, whose cross-ratio is $z_e = z$, where $e = [0\infty]$. The left triangle tick-mark is located at i, while the right one is located at ie^{-z}, after the homothety sending the triangle into the standard triangle. Taking in account that the orientation of the edge e is up-side one derives that t_e is the signed hyperbolic distance between i and $e^{-z_e}i$, which is z_e.

Proposition 25. *The map $\mathbf{t}_\Gamma : \mathcal{T}_g^s \to \mathbb{R}^{E_\Gamma}$ given by $t_\Gamma(S) = (t_e)_{e\in E_\Gamma}$ is a homeomorphism onto the linear subspace $\mathbb{R}^{E_\Gamma/F_\Gamma} \subset \mathbb{R}^{E_\Gamma}$ given by equations:*

$$t_\gamma := \sum_{k=1}^{n} t_{e_k} = 0,$$

for all left-hand-turn closed paths $\gamma \in F_\Gamma$, which is expressed as a cyclic chain of edges $e_1, ..., e_n$.

Remark 34. Notice that there are exactly s left-hand-turn closed paths, which lead to s independent equations hence the subspace $\mathbb{R}^{E_\Gamma/F_\Gamma}$ from above is of dimension $6g - 6 + 2s$.

Proof. The map $\mathbf{t}_\Gamma$ is continuous, and it suffices to define an explicit inverse for it. Let Γ be a trivalent fatgraph whose edges are labeled by real numbers $\mathbf{r} = (r_e)_{e\in E_\Gamma}$. We want to paste one copy Δ_v of the standard marked ideal triangle on each vertex v of Γ and glue together by isometries these triangles according to the edges connections. Since the edges of an ideal triangle are of infinite length we have the freedom to use arbitrary translations along these geodesics when gluing together adjacent sides. If $e = [vw]$ is an edge of Γ then one can associate a real number $t_e \in \mathbb{R}$ as follows. There are two tick-marks, namely t_v and t_w on the common side of Δ_v and Δ_w. We denote by t_e the amount needed for translating t_v into t_w according to the orientation inherited as a boundary of Δ_v. Given now the collection of real numbers $\mathbf{r}$ we can construct unambiguously our Riemann surface $S(\Gamma, \mathbf{r})$, which moreover has the property that $t_\Gamma(S(\Gamma, \mathbf{r})) = \mathbf{r}$. Furthermore it is sufficient now to check whenever this constructions yields a complete Riemann surfaces. The completeness at the puncture determined by the left-hand-turn path γ is equivalent to the condition $t_\gamma = 0$, and hence the claim. The cusps of $S(\Gamma)$ are in bijection with the left-hand-turn paths in Γ, and the triangulation of $S(\Gamma)$ obtained by our construction corresponds to Γ.

The Fuchsian group associated to Γ and $\mathbf{r}$

The surface $S(\Gamma, \mathbf{r})$ is uniformized by a Fuchsian group $G = G(\Gamma, \mathbf{r}) \subset \mathrm{PSL}(2, \mathbb{R})$, i.e. $S(\Gamma, \mathbf{r}) = \mathcal{D}/G(\Gamma, \mathbf{r})$. We can explicitly determine the generators of the Fuchsian group, as follows.

We have natural isomorphisms between the fundamental group $\pi_1(S(\Gamma, \mathbf{r})) \cong \pi_1(\Gamma^t) \cong \pi_1(\Gamma)$. Any path γ in Γ is a cyclic sequence of adjacent directed edges $e_1, e_2, e_3, ..., e_n$, where e_i and e_{i+1} have the vertex v_i in common. We insert between e_i and e_{i+1} the symbol lt if e_i, e_{i+1} is a left-hand-turn, the symbol rt if it is a right-hand-turn and no symbol otherwise (i.e. when e_{i+1} is e_i with the opposite orientation). Assume now that we have a Riemann surface whose coordinates are $\mathbf{t}_\Gamma(S) = \mathbf{r}$. We define then a representation $\rho_{\mathbf{r}} : \Pi_1(\Gamma) \to \mathrm{PSL}(2, \mathbb{R})$ of the path groupoid $\Pi_1(\Gamma)$ by the formulas:

$$\rho_{\mathbf{r}}(e) = \begin{pmatrix} 0 & e^{\frac{r_e}{2}} \\ -e^{-\frac{r_e}{2}} & 0 \end{pmatrix}, \text{ and } \rho_{\mathbf{r}}(lt) = \rho_{\mathbf{r}}(rt)^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

This is indeed well-defined since $\rho_{\mathbf{r}}(e)^2 = -1 = 1 \in \mathrm{PSL}(2, \mathbb{R})$, and hence the orientation of the edge does not matter, and $\rho_{\mathbf{r}}(lt)^3 = \rho_{\mathbf{r}}(rt)^3 = 1$. Furthermore the fundamental group $\pi_1(\Gamma)$ is a subgroup of $\Pi_1(\Gamma)$.

Proposition 26. *The Fuchsian group $G(\Gamma, \mathbf{r})$ is $\rho_{\mathbf{r}}(\pi_1(\Gamma)) \subset \mathrm{PSL}(2, \mathbb{R})$.*

Proof. We can begin doing the pasting without leaving the hyperbolic plane, until we get a polygon P, together with a side pairing. We may think of each triangle as having a white face and a black face, and build the polygon P such that all the triangles have white face up. We attach to each side pairing (s_i, s_j) an orientation preserving isometry A_{ij}, such that $A_{ij}(s_i) = s_j$, A_{ij} sends tick-marks into the tick-marks shifted by r_e, and $P \cap A_{ij}(P) = \emptyset$. Denote by G the subgroup of $\mathrm{Isom}^+(\mathcal{D})$ generated by all the side-pairing transformations. In order to apply the Poincaré Theorem all the vertex-cycle transformations must be parabolic. This amounts to ask that for every left-hand-turn closed path γ we have $t_\gamma = 0$. Then by the Poincaré theorem G is a discrete group of isometries with P as its fundamental domain and $\mathcal{D}/G$ is the complete hyperbolic Riemann surface $S(\Gamma, \mathbf{r})$.

We need now the explicit form of the matrices A_{ij}. We obtain them by composing the isometries sending a marked triangle into the adjacent one, in a suitable chain of triangles, where consecutive ones have a common edge. If e is such an edge we remark that $\rho_{\mathbf{r}}(e)$ do the job we want, because it sends the triangle $[-1, 0, \infty]$ into $[e^{r_e}, \infty, 0]$. Moreover the quadrilateral $[-1, 0, e^{r_e}, \infty]$, with this decomposition into two triangles, has associated the cross-ratio r_e. We need next to use $\rho_{\mathbf{r}}(lt)$ which permutes counter-clockwise the tick-marks and the vertices $-1, 0$ and ∞ of the ideal triangle. Then one identifies the matrices A_{ij} with the images of the closed paths by $\rho_{\mathbf{r}}$.

Remark 35. We observe that the left-hand-turn paths are preserved under an isomorphism of graphs which preserves the cyclic orientation at each vertex. Thus any automorphism of the fatgraph Γ induces an automorphism of $S(\Gamma)$.

10.3 Coordinates on the Teichmüller space of surfaces with geodesic boundary

Set $\mathcal{T}_{g,s;or}$ for the Teichmüller space of surfaces of genus g with s *oriented* boundary components. Here *or* denotes the choice of one orientation for each of the boundary components. Since the surface has a canonical orientation, we can set unambiguously $or : \{1, 2, ..., s\} \to \mathbb{Z}/2\mathbb{Z}$ by assigning $or(j) = +1$ if the orientation of the j-th component agrees with that of the surface and $or(j) = -1$, otherwise. We suppose that each boundary component is a geodesic in the hyperbolic metric, and possibly a cusp (hence in some sense this space is slightly completed). Let Γ be a fatgraph with the property that Γ^t is a surface of genus g with s boundary components and let S denote the surface Γ^t endowed with a hyperbolic structure, for which the boundary is geodesic. Assume that, in this metric, the boundary geodesics b_j have length l_j.

Consider the restriction of the hyperbolic metric to $int(\Gamma^t) = \Gamma^o$. Then Γ^o is canonically homeomorphic to the punctured surface $\Gamma^c - \{p_1, ..., p_s\}$. In particular there is a canonically induced hyperbolic metric on $\Gamma^c - \{p_1, ..., p_s\}$, which we denote by S^*. Moreover this metric is not complete at the punctures p_j. Suppose that the punctures p_j corresponds to the left-hand-turn closed paths γ_j, or equivalently the boundary components geodesics b_j, of length l_j. Assume that we have an ideal triangulation of S^* by geodesic simplices, whose ideal vertices are the punctures p_j. Then the holonomy of the hyperbolic structure around the vertex p_j is a non-trivial, and it can be calculated in the following way (see [26], Prop.3.4.18, p.148). Consider a geodesic edge α entering the puncture and a point $p \in \alpha$. Then the geodesic spinning around p_j in the positive direction (according to the orientation of the boundary circle) is intersecting again α a first time in the point $h_{p_j}(p)$. The hyperbolic distance between the points p and $h_{p_j}(p)$ is the length l_j of the boundary circle in the first metric. Moreover the point $h_{p_j}(p)$ lies in the ray determined by p and the puncture p_j. Notice that if we had chose the loop encircling the puncture to go in opposite direction then the iterations $h_{p_j}(p)$ would have gone faraway from the puncture, and the length would have been given the negative sign. Set therefore l_j^o* for the signed length.

We construct as above the geodesic ideal triangulation $\Delta(\Gamma)$ of the non-complete hyperbolic punctured surface S^*. We can therefore compute the holonomy map using the thick-marks on some edge abutting to the puncture p_j. It is immediately that the the holonomy displacement on this edge is given by t_{γ_j}, where γ_j is the left-hand-turn closed path corresponding to this puncture. In particular we derive that:

$$|t_{\gamma_j}| = l_j, \text{ for all } j \in \{1, 2, ..., s\}.$$

Using the method from the previous section we know how to associate to any edge e of Γ a real number $t_e = t_e(S^*)$ measuring the shift between two ideal triangles in the geodesic triangulation of the surface S^*.

Proposition 27. *The map $\mathbf{t}_\Gamma : \mathcal{T}_{g,s;or} \to \mathbb{R}^{E_\Gamma}$ given by $t_\Gamma(S) = (t_e)_{e \in E_\Gamma}$ is a homeomorphism.*

Proof. The construction of an inverse map proceeds as above. Given $\mathbf{r} \in \mathbb{R}^{E_\Gamma}$ we construct a non-complete hyperbolic surface S^* with s punctures with the given parameters, by means of gluing ideal triangles. As shown in ([26], Prop. 3.4.21, p.150) we can complete this hyperbolic structure to a surface with geodesic boundary S, such that $int(S) = S^*$. Further if $t_{\gamma_j} > 0$, then we assign the orientation of γ_j for the boundary component b_j, otherwise we assign the reverse orientation. When $t_{\gamma_j} = 0$ it means that we have a cusp at p_j.

Remark 36. The two points of $\mathcal{T}_{g,s;or}$ given by the same hyperbolic structure on the surface $\Sigma_{g,s}$ but with distinct orientations of some boundary components lie in the same connected component. Nevertheless the previous formulas shows that a path connecting them must pass through the points of $\mathcal{T}_{g,s;or}$ corresponding to surfaces having a cusp at the respective puncture.

Set $\mathcal{T}_{g,s}$ for the Teichmüller space of surfaces of genus g with s *non-oriented* boundary components, i.e. hyperbolic metrics for which the boundary components are geodesic. There is a simple way to recover coordinates on $\mathcal{T}_{g,s}$ from its oriented version. Let $\psi : \mathbb{R}^{E_\Gamma} \to \mathbb{R}^{F_\Gamma}$ be the map $\psi(\mathbf{t}) = (t_{\gamma_i})_{\gamma_i \in F_\Gamma}$. Choose a projector $\psi^* : \mathbb{R}^{E_\Gamma} \to \ker \psi = \mathbb{R}^{E_\Gamma / F_\Gamma}$, and set $\iota_{|.|} : \mathbb{R}^{F_\Gamma} \to \mathbb{R}^{F_\Gamma}$ for the map given on coordinates by $\iota_{|.|}(y_j)_{j=1,\sharp F_\Gamma} = (|y_j|)_{j=1,\sharp F_\Gamma}$. Then $\mathcal{T}_{g,s}$ is the quotient by the $(\mathbb{Z}/2\mathbb{Z})^{F_\Gamma}$-action on $\mathcal{T}_{g,s;or}$ which changes the orientation of the boundary components.

Proposition 28. *We have a homeomorphism $\mathbf{t}_\Gamma : \mathcal{T}_{g,s} \to \mathbb{R}^{6g-6+2s} \oplus \mathbb{R}^s$, which is induced from the second line of the following commutative diagram:*

$$\begin{tikzcd}
\mathcal{T}_{g,s;or} \arrow[r, "(\psi^*\oplus\psi)\circ\mathbf{t}_\Gamma"] \arrow[d] & \mathbb{R}^{E_\Gamma/F_\Gamma}\oplus\mathbb{R}^{F_\Gamma} \arrow[d, "id\oplus\iota_{|.|}"] \\
\mathcal{T}_{g,s} \arrow[r] & \mathbb{R}^{E_\Gamma/F_\Gamma}\oplus\mathbb{R}_+^{F_\Gamma} \\
\mathcal{T}_g^s \arrow[r] \arrow[u] & \mathbb{R}^{E_\Gamma/F_\Gamma} \arrow[u, "id\oplus 0"]
\end{tikzcd}$$

Remark 37. Observe that the embedding $\mathcal{T}_g^s \hookrightarrow \mathcal{T}_{g,s}$ given in terms of coordinates by adding on the right a string of zeroes lifts to an embedding $\mathcal{T}_g^s \hookrightarrow \mathcal{T}_{g,s;or}$.

Putting together the results of the last two sections we derive that:

Proposition 29. *The map $\mathbf{t}_\Gamma : \mathcal{T}_{g,n;or}^s \to \mathbb{R}^{E_\Gamma}$ given by $t_\Gamma(S) = (t_e)_{e\in E_\Gamma}$ is a homeomorphism of the Teichmüller space of surfaces of genus g with n oriented boundary components and s punctures onto the linear subspace $\mathbb{R}^{E_\Gamma/F^*\Gamma}$ of dimension $6g - 6 + 3n + 2s$ given by the equations: $t_{\gamma_j} = 0$, for those left-hand-turn closed paths γ_j corresponding to the punctures, $\gamma_j \in F_\Gamma^* \subset F_\Gamma$.*

Remark 38. W.Thurston associated to an ideal triangulation a system of shearing coordinates for the Teichmüller space in mid eighties (see [27]) and from a slightly different perspective in Penner's treatment of the decorated Teichmüller spaces ([22]). The systematic study of such coordinates appeared later in the papers of F.Bonahon [2] and V.Fock unraveled in [5] the elementary aspects of this theory which lead him further to the quantification of the Teichmüller space.

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11

The interplay between mapping class groups and Teichmüller spaces

11.1 Mapping class groups acting on Teichmüller spaces

There is a close relation between mapping class groups and Teichmüller spaces. The Dehn-Nielsen-Baer theorem provides an identification between $\mathrm{Mod}(\Sigma)$ and $\mathrm{Out}^+(\pi_1\Sigma)$. Then mapping class group acts by left composition on the space $\mathcal{T}(\Sigma)$ which is a space of group representations, up to conjugacy:

$$\mathcal{T}(\Sigma) = \mathrm{Hom}^+_{f,d}\,(\pi_1\Sigma, \mathrm{PSL}(2,\mathbb{R}))\,/\,\mathrm{PSL}(2,\mathbb{R})$$

Specifically, this action is given by

$$(\varphi, [\rho]) \longrightarrow [\rho\circ\varphi^{-1}]$$

Moreover, $\mathrm{Mod}(\Sigma)$ acts by real analytic homeomorphisms. This action is important in understanding both the algebraic structure of the mapping class group using the geometry of the Teichmüller space, because of the following basic result going back to F.Klein and R.Fricke.

Proposition 30 (Fricke-Klein 1889, Kravetz 1959). $\mathrm{Mod}(\Sigma)$ *acts properly discontinuously on* $\mathcal{T}(\Sigma)$.

Proof. Let assume that there exist a sequence $\varphi_n \in \mathrm{Mod}\,\Sigma$ so that there exist two compacts C_1, C_2 in the Teichmüller space with the property $\varphi_n(C_1)\cap C_2 \neq \emptyset$ for all n. Then there exists a convergent sequence of points $z_n \to z \in \mathcal{T}(\Sigma)$ so that $\varphi_n(z_n)$ also converges to some point $w \in \mathcal{T}(\Sigma)$. Thus $\varphi_n^{-1}\varphi_{n-1}(z_n) \to z$. We will show that if $\xi_n \in \mathrm{Mod}\,\Sigma$ has the property that $\xi_n z_n \to z$ then $\xi_n = \mathbb{1}$ for large enough n.

This is a consequence of the following facts:

1. If Γ is a Fuchsian group then the set

$$A(\Gamma) = \{\mathrm{Tr}(\gamma); \gamma \in \Gamma \subset \mathrm{PSL}(2,\mathbb{R})\} \subset \mathbb{R}_+$$

is mapped by the function $\cosh\frac{1}{2}(x)$ bijectively into the marked set of lengths of geodesics of the surface $\mathcal{H}/\Gamma$ (indexed by elements of Γ). Moreover, these sets are discrete.
2. If $\xi \in \mathrm{Mod}(\Sigma)$ and $\Gamma = \rho(\pi_1\Sigma)$ is a Fuchsian group then the marked set $A(\xi\Gamma)$ is obtained from the marked set $A(\Gamma)$ by a permutation of its elements.
3. The regular functions $\mathrm{tr}(\rho(\gamma))$, $\gamma \in \pi_1\Sigma$, viewed as functions $\mathcal{T}(\Sigma) \to \mathbb{R}$ are generating a polynomial algebra which is finitely generated. The proof is based on the identity:
$$\mathrm{tr}(x)\,\mathrm{tr}(y) = \mathrm{tr}(xy) + \mathrm{tr}(xy^{-1}).$$
4. If $\xi_n z_n \to z$ then for large n
$$A(\xi_n\Gamma_{zn}) \text{ and } A(\Gamma_s) \text{ agree on their first } N \text{ items}$$
It N is large enough in order that all generators of the algebra above are contained among the first N items then we find that $A(\xi_n\Gamma_{zn}) = A(\Gamma_z)$. Since ξ_n acts as a permutation on the marked sets of geodesics we derive that the permutation is the identity.
5. Two hyperbolic structures on a surface having the same marked lengths of geodesics are isometric. In fact, if the traces of two discrete faithful representations coincide i.e. $\mathrm{tr}(\rho(\gamma) = \mathrm{tr}(\rho'(\gamma))$ for any $\gamma \in \pi_1(\Sigma)$ then the representations are conjugate.

Remark 39. 1. The $\mathrm{Mod}(\Sigma_g)$-action on the Teichmüller space is *effective* if $g \geq 3$. When $g = 1, 2$ the hyperelliptic involution acts trivially on $\mathcal{T}(\Sigma)$.
2. The quotient $\mathcal{T}(\Sigma)/\mathrm{Mod}(\Sigma)$ is naturally a complex space with orbifold singularities (at points where the $\mathrm{Mod}(\Sigma)$ action is not free). However, one knows that all stabilizers should be finite. In this respect the moduli space $\mathcal{M}(\Sigma) = \mathcal{T}(\Sigma)/\mathrm{Mod}(\Sigma)$ plays the role of a *classifying space for the mapping class group*. For instance, we have an isomorphism
$$H^*(\mathcal{M}(\Sigma); \mathbb{Q}) \simeq H^*(\mathrm{Mod}(\Sigma); \mathbb{Q})$$
3. Since $\mathcal{T}(\Sigma)$ is a topological cell each torsion element of $\mathrm{Mod}(\Sigma)$ should fix a non-empty set. In particular, any periodic mapping class contains a periodic homeomorphism which is a conformal homeomorphism for some complex structure on Σ.

References

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2. S.Kravetz, *On the geometry of Teichmüller spaces and the structure of their modular groups*, Ann. Acad. Sci. Fenn. 278 (1959), 1–35.

11.2 Stabilizers of the mapping class group action

The action of $\mathrm{Mod}(\Sigma)$ on $\mathcal{T}(\Sigma)$ is properly discontinuous and hence it has finite stabilizers. A point p in $\mathcal{T}(\Sigma)$ corresponds to a class of marked Riemann surface $p = [S]$, and we can identify the stabilizer $\mathrm{Mod}(\Sigma)_p$ of the point p, as follows:

$$\mathrm{Mod}(\Sigma)_p = \{\varphi \in \mathrm{Mod}(\Sigma) \text{ such that } [\varphi S] = [S]\}$$

Moreover, S is defined by the holonomy map $\rho_S : \pi_1\Sigma \to \mathrm{PSL}(2, \mathbb{R})$ and so we have $\rho_{\varphi S} = \varphi \circ \rho_S$, where φ is interpreted now as an element of $\mathrm{Out}^+(\pi_1\Sigma)$. Since the marked surfaces determined by $\rho_{\varphi S}$ and ρ_S are the same they should be obtained by means of a conjugation within $\mathrm{PSL}(2, \mathbb{R})$ i.e. there exists $\lambda = \lambda_\varphi \in \mathrm{PSL}(2, \mathbb{R})$ so that

$$\rho_{\varphi S} = \lambda_\varphi \rho_S \lambda_\varphi^{-1}$$

In particular, λ_φ belongs to the normalizer of the Fuchsian group $\rho_S(\pi_1 S)$ and it is immediate that the map

$$\lambda : \mathrm{Mod}(\Sigma)_p \longrightarrow N\left(\rho_S(\pi_1\Sigma)\right)/\rho_S(\pi_1\Sigma)$$

is a group homomorphism. Actually, we have a more precise result:

Proposition 31. *The stabilizer of the class of the marked Riemann surface* $[S]$ *is given by*

$$\mathrm{Mod}(\Sigma)_{p=[S]} = \mathrm{Aut}(S)$$

where $Aut(S)$ *are the conformal (i.e. holomorphic) automorphism group of* S*.*

In fact, any element of $\mathcal{N}(\Gamma)/\Gamma$, Γ Fuchsian group corresponds to an automorphism of the Riemann surface (see the section 9.8).

Corollary 18. *For a generic Riemann surface* S *we have* $\mathrm{Aut}(S) = \{\mathbb{1}\}$.

Remark 40. It is known that, if the genus of Σ is $g \geq 4$, then the local structure of $\mathcal{T}(\Sigma)/\mathrm{Mod}(\Sigma)$ around $p \in \mathcal{T}(\Sigma)/\mathrm{Mod}(\Sigma)$ is described by the quotient $\mathbb{R}^{6g-6}/F_p$, where $F_p \subset \mathrm{GL}(6g-6)$ is a finite group, which is the image of a faithful linear representation $\mathrm{Aut}(S) \to \mathrm{GL}(6g-6)$ (where $[S] = p$).

In particular, the point p is smooth in the quotient iff S has no automorphisms.

A more elaborate analysis shows that the space $\mathcal{T}(\Sigma)/\mathrm{Mod}(\Sigma)$ is singular at the points when S has automorphisms (as shown by E.Rauch in 1962) for $g \geq 4$. For $g = 2$ there is only one singular point, corresponding to the Riemann surface given by the equation:

$$y^2 = x^5 - 1$$

which has additional symmetries with respect to the rest of Riemann surfaces having only the hyperelliptic involution automorphism. For $g = 3$ the hyperelliptic locus consists of smooth points.

Furthermore it is known that there exists a finite index subgroup of $\mathrm{Mod}(\Sigma)$ which acts freely on $\mathcal{J}(\Sigma)$. A quantitative estimate of the index follows from the following result due to J. P. Serre (1958):

Proposition 32. *If* $\varphi \in \mathrm{Mod}(\Sigma)_{[S]}$ *is an automorphism of the Riemann surface* S *and*

$$\varphi_* : H_1(\Sigma, \mathbb{Z}/\ell\mathbb{Z}) \longrightarrow H_1(\Sigma, \mathbb{Z}/\ell\mathbb{Z})$$

is the identity for some $\ell \geq 3$ *then* $\varphi = \mathbb{1}$. *In particular* $\ker(\mathrm{Mod}(\Sigma) \to Aut(H_1(\Sigma;\mathbb{Z}/\ell\mathbb{Z})) \cong Sp(2g,\ell))$ *acts freely on* $\mathcal{T}(\Sigma)$ *for any* $\ell \geq 3$.

11.3 The Ptolemy modular groupoid

The modular groupoid was considered by Mosher in his thesis and further as a key ingredient in [18, 19], it is implicit in Harer's paper on the arc complex (see [12]) and then studied by Penner (see [22, 23]; notice that the correct definition is that from [23]) who introduced also the terminology.

Recall that a groupoid is a category whose morphisms are invertible, such that between any two objects there is at least one morphism. The morphisms from an object to itself form a group (the group associated to the groupoid).

Remark 41. Suppose that we have an action of a group G on a set M. We associate a groupoid $\mathcal{G}(G, M)$ as follows: its objects are the G-orbits on M, and the morphisms are the G-orbits of the diagonal action on $M \times M$. If the initial action was free then G embeds in $\mathcal{G}(G, M)$ as the automorphisms group of any object.

Assume that we have an ideal triangulation $\Delta(\Gamma)$ of a surface Σ_g^s. If e is an edge shared by the triangles Δ_v and Δ_w of the triangulation then we change the triangulation by excising the edge e and replacing it by the other diagonal of the quadrilateral $\Delta_v \cup \Delta_w$, as in figure 11.1. This operation $F[e]$ was called flip in [5] or elementary by Mosher and Penner.

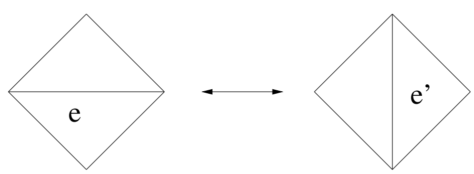


Fig. 11.1. The flip

Let $\mathcal{IT}(\Sigma_g^s))$ denote the set of isotopy classes of ideal triangulations of Σ_g^s. The reduced *Ptolemy groupoid* $\overline{P_g^s}$ is the groupoid generated by the flips action on $\mathcal{I}T(\Sigma_g^s)$). Specifically its elements are classes of sequences $\Delta_0, \Delta_1, ..., \Delta_m$, where Δ_{j+1} is obtained from Δ_j by using a flip. Two sequences $\Delta_0, ..., \Delta_m$ and $\Delta'_0, ..., \Delta'_n$ are equivalent if their initial and final terms coincide i.e. there exists a homeomorphism φ preserving the punctures such that $\varphi(\Delta_0) \cong \Delta'_0$ and $\varphi(\Delta_m) \cong \Delta'_n$, where $\cong$ denotes the isotopy equivalence. Notice that any two (isotopy classes of) ideal triangulations are connected by a chain of flips (see [13] for an elementary proof), and hence $\overline{P_g^s}$ is indeed an groupoid. Moreover $\overline{P_g^s}$ is the groupoid $\mathcal{G}(\mathcal{M}_g^s, \mathcal{IT}(\Sigma_g^s))$ associated to the obvious action of the mapping class group $\mathcal{M}_g^s$ on the set of isotopy classes of ideal triangulations $\mathcal{IT}(\Sigma_g^s)$). One problem in considering $\overline{P_g^s}$ is that the action of $\mathcal{M}_g^s$ on $\mathcal{IT}(\Sigma_g^s)$) is *not free* but there is a simple way to remedy it. For instance in [18, 19] one adds the extra structure coming from fixing an *oriented arc* of the ideal triangulation. A second problem is that we want that the mapping class group action on the Teichmüller space extends to a groupoid action.

Consider now an ideal triangulation $\Delta = \Delta(\Gamma)$, where Γ is its dual fatgraph. A labelling of Δ is a numerotation of its edges $\sigma_\Gamma : E_\Gamma \to \{1, 2, ..., \sharp E_\Gamma\}$. Set now $\mathcal{LIT}(\Sigma_g^s)$) for the set of labeled ideal triangulations. The Ptolemy groupoid P_g^s of the punctured surface Σ_g^s is the groupoid generated by flips on $\mathcal{LIT}(\Sigma_g^s)$). The flip $F[e]$ associated to the edge $e \in E_\Gamma$ acts on the labellings in the obvious way:

$$\sigma_{F[e](\Gamma)}(f) = \begin{cases} \sigma_\Gamma(f), & \text{if } f \neq e' = F[e](e) \\ \sigma_\Gamma(e), & \text{if } f = e', \end{cases}$$

According to ([23] Lemma 1.2.b), if $2g-2+s \geq 2$ then any two labeled ideal triangulations are connected by a chain of flips, and thus P_g^s is indeed a groupoid. Moreover, this allows us to identify P_g^s with $\mathcal{G}(\mathcal{M}_g^s, \mathcal{LIT}(\Sigma_g^s))$.

Remark 42. In the remaining cases, namely Σ_0^3 and Σ_1^1, the flips are not acting transitively on the set of labeled ideal triangulations. In this situation an appropriate labelling consist in an oriented arc, as in [18]. The Ptolemy groupoid associate to this labeling has the right properties, and it acts on the Teichmüller space.

Proposition 33. *We have an exact sequence*

$$1 \to \mathcal{S}_{6g-6+3s} \to P_g^s \to \overline{P_g^s} \to 1,$$

where $\mathcal{S}_n$ denotes the symmetric group on n letters. Notice that $P_1^1 = \overline{P_1}^1$. If $(g, s) \neq (1, 1)$ then $\mathcal{M}_g^s$ naturally embeds in P_g^s as the group associated to the groupoid.

Proof. The first part is obvious. The following result is due to Penner ([23],Thm.1.3):

Lemma 48. *If $(g, s) \neq (1, 1)$ then $\mathcal{M}_g^s$ acts freely on $\mathcal{LIT}(\Sigma_g^s)$.*

Proof. A homeomorphism keeping invariant a labeled ideal triangulation either preserves the orientation of each arc or else it reverses the orientation of all arcs.

In fact once the orientation of an arc lying in some triangle is preserved, the orientation of the other boundary arcs of the triangle must also be preserved. Further in the first situation either the surface is Σ_0^3 (when $\mathcal{M}_0^3 = 1$) or else each triangle is determined by its 1-skeleton, and the Alexander trick shows that the homeomorphism is isotopic to identity. In the second case we have to prove that $(g, s) = (1, 1)$. Since the arcs cannot have distinct endpoints we have $s = 1$. Let Δ_1 be an oriented triangle and $D \subset \Delta_1$ be a 2-disk which is a slight retraction of Δ_1 into its interior. The image D' of D cannot lie within Δ_1 because the homeomorphism is globally orientation preserving while the orientation of the boundary of D' is opposite to that of $\partial\Delta$. Thus D' lies outside Δ_1 and the region between $\partial D'$ and $\partial\Delta_1$ is an annulus, so the complementary of Δ_1 consists of one triangle. Therefore $g = 1$.

Remark 43. The punctured torus Σ_1^1 has an automorphism which reverse the orientation of each of the three ideal arcs.

The case of the punctured torus is settled by the following:

Proposition 34. *Let $\Delta_{st} = \{\alpha_1, \alpha_2, \alpha_3\}$, where $\alpha_1 = (1,0), \alpha_2 = (1,1), \alpha_3 = (0,1)$ be the standard labeled ideal triangulation of the punctured torus $\Sigma_1^1 = \mathbb{R}^2/\mathbb{Z}^2 - \{0\}$.*

1. *If $\Delta = \{\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \alpha_{\sigma(3)}\}$ is flip equivalent to Δ_{st} then σ is the identity.*
2. *A mapping class which leaves invariant Δ_{st} is either identity or $-id \in SL(2,\mathbb{Z}) = \mathcal{M}_1^1$.*
3. *Let $\Delta = \{\gamma_1, \gamma_2, \gamma_3\}$ be an arbitrary ideal triangulation. Then there exists an unique $\sigma(\Delta) \in \mathcal{S}_3$ such that Δ is flip equivalent with the labeled diagram $\{\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \alpha_{\sigma(3)}\}$.*
4. *In particular if $\Delta = \varphi(\Delta_{st})$ then we obtain a group homomorphism $\sigma : SL(2,\mathbb{Z}) \to \mathcal{S}_3$, given by $\sigma(\varphi) = \sigma(\varphi(\Delta_{st}))$, whose values can be computed from:*

$$\sigma\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = (23), \quad \sigma\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = (12), \quad \sigma\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (13).$$

We need therefore another labeling for Σ_1^1, which amounts to fix a *distinguished oriented edge (d.o.e.)* of the triangulation. The objects acted upon flips are therefore pairs (Δ, e), where e is the d.o.e. of Δ. A flip acts on the set of labeled ideal triangulations with d.o.e. as follows. If the flip leaves e invariant then the new d.o.e. is the old one. Otherwise the flip under consideration is $F[e]$, and the new d.o.e. will be the image e' of e, oriented so that the frame (e, e') at their intersection point is positive with respect to the surface orientation. The groupoid Pt_g^s generated by flips on (labeled) ideal triangulations with d.o.e. of is called the *extended Ptolemy groupoid*. Since any edge permutation is a product of flips (when $(g, s) \neq 1$) it follows that any two labeled triangulations with d.o.e. can be connected by a chain of flips.

The case of the punctured torus is subjected to caution again: it is more convenient to define the groupoid Pt_1^1 as that generated by iterated compositions of flips on the standard (labeled or not) ideal triangulation Δ_{st} of Σ_1^1 with a fixed d.o.e., for instance α_1. In fact proposition 34 implies that there are three distinct orbits of the flips on triangulations with d.o.e., according to the the position of the d.o.e. within Δ_{st}.

Remark 44. For all (g, s) we have an exact sequence:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathsf{Pt}_g^s \to P_g^s \to 1.$$

Moreover $\mathcal{M}_g^s \to P_g^s$ lifts to an embedding $\mathcal{M}_g^s \hookrightarrow \mathsf{Pt}_g^s$.

Remark 45. We can define the groupoid $\overline{\mathsf{Pt}_g^s}$ by considering flips on ideal triangulations with d.o.e. without labellings.

Remark 46. The kernel of the map $\mathcal{M}_1^1 \to P_1^1$ is the group of order two generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Therefore any (faithful) representation of P_1^1 induces a (faithful) representation of $\mathrm{PSL}(2, \mathbb{Z})$.

Remark 47. One reason to consider P_g^s instead of $\overline{P_g^s}$ is that P_g^s acts on the Teichmüller space while $\overline{P_g^s}$ does not. The other reason is that $\mathcal{M}_g^s$ injects into P_g^s (if $(g, s) \neq (1, 1)$). The kernel of $\mathcal{M}_g^s \to \overline{P_g^s}$ is the image of the automorphism group $Aut(\Gamma)$ in $\mathcal{M}_g^s$.

Proof. An automorphism of Γ is a combinatorial automorphism which preserves the cyclic orientation at each vertex. Notice that an element of $Aut(\Gamma)$ induces a homeomorphism of Γ^t and hence an element of $\mathcal{M}_g^s$. Now, if φ is in the kernel then φ is described by a permutation of the edges i.e. an element of $\varphi_* \in \mathcal{S}_{\sharp E_\Gamma}$. One can assume that the orientations of all arcs are preserved by φ when $(g, s) \neq (1, 1)$. Then φ_* completely determines φ, by the Alexander trick. Further φ induces an element of $Aut(\Gamma)$ whose image in $\mathcal{S}_{\sharp E_\Gamma}$ is precisely φ_*. This establishes the claim. Notice that the map $Aut(\Gamma) \to \mathcal{S}_{\sharp E_\Gamma}$ is injective for most but not for all fatgraphs Γ. The fatgraphs Γ for which the map $Aut(\Gamma) \to \mathcal{S}_{\sharp E_\Gamma}$ fails to be injective are described in [20].

We can state now a presentation for Pt_g^s which is basically due to Penner ([23]):

Proposition 35. Pt_g^s *is generated by the flips $F[e]$ on the edges. The relations are:*

1. *Set J for the change of orientation of the d.o.e. Then*

$$F[F[e]e]F[e] = \begin{cases} 1, & \text{if } e \text{ is not the d.o.e.} \\ J, & \text{if } e \text{ is the d.o.e.} \end{cases}$$

2. $J^2 = 1$.

3. *Consider the pentagon from picture 11.2, and* $F[e_j]$ *be the flips on the dotted edges. Let* $\tau_{(12)}$ *denote the transposition interchanging the labels of the two edges* e_1 *and* f_1 *from the initial triangulation. Then we have:*

$$F[e_1]F[e_2]F[e_3]F[e_4]F[e_5] = \begin{cases} J\tau_{(12)}, & \text{if } e_1 \text{ is not the d.o.e.} \\ \tau_{(12)}, & \text{if } e_1 \text{ is the d.o.e.} \end{cases}$$

The action of $\tau_{(12)}$ *on triangulations with d.o.e. is at follows: if none of the permuted edges* e, f *is the d.o.e. then* $\tau_{(12)}$ *leaves the d.o.e. unchanged. If the d.o.e. is one of the permuted edges, say* e*, then the new d.o.e. is* f *oriented such that* e *(with the former d.o.e. orientation) and* f *with the given d.o.e. orientation form a positive frame on the surface. Notice that* $[F[e_1]F[e_2]F[e_3]F[e_4]F[e_5] = \tau_{(12)}$ *even if* f_1 *is the d.o.e.*

4. *If* e *and* f *are disjoint edges then* $F[e]F[f] = F[f]F[e]$.

5. *The relations in a* $\mathbb{Z}/2\mathbb{Z}$ *extension of the symmetric group, expressed in terms of flips. To be more specific, les us assume that the edges are labeled and the d.o.e. is labelled* 0*. Then we have:*

$$\tau_{(0i)}^2 = J,\ \tau_{(ij)}^2 = 1,\ \text{if } i,j \neq 0,\ \tau_{(st)}\tau_{(mn)} = \tau_{(mn)}\tau_{(st)} \text{ if } \{m,n\} \cap \{s,t\} = \emptyset,$$

$$\tau_{(st)}\tau_{(tv)}\tau_{(st)} = \tau_{(tv)}\tau_{(st)}\tau_{(tv)},\ \text{if } s,t,v \text{ are distinct.}$$

6. $F[\tau(e)]\tau F[e] = \tau$, *for any label transposition* τ *(expressed as a product of flips as above), which says that the symmetric group is a normal subgroupoid of* P_g^s.

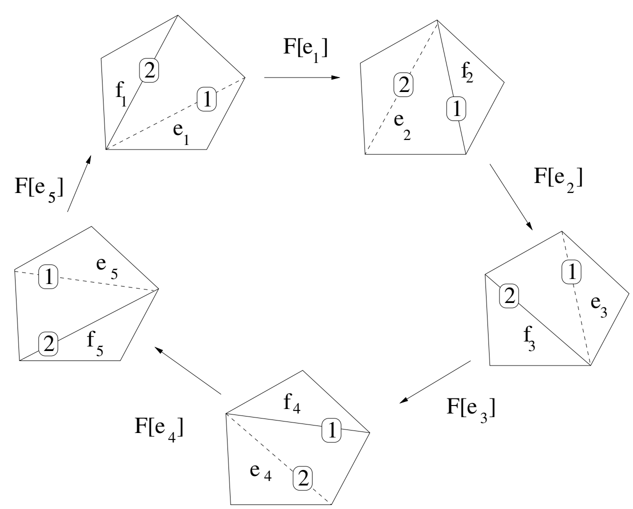


Fig. 11.2. The pentagon relation

Proof. We analyze first the case where labellings are absent:

Lemma 49. *$\overline{P_g^s}$ is generated by the flips on edges $F[e]$. The relations are:*

1. *$F[e]^2 = 1$, which is a fancy way to write that the composition of the flip on $F[e](e)$ with the flip on e is trivial.*
2. *$F[e_1]F[e_2]F[e_3]F[e_4]F[e_5] = 1$, where $F[e_i]$ are the flips considered in the picture 11.2.*
3. *Flips on two disjoint edges commute each other.*

Proof. This result is due to Harer (see [12]). It was further exploited by Penner ([22, 23]).

The complete presentation is now a consequence of the two exact sequences from proposition 33 and remark 44, relating $\overline{P_g^s}$, P_g^s and P_g^s.

Remark 48. By setting $J = 1$ above we find the presentation of P_g^s, with which we will be mostly concerned in the sequel.

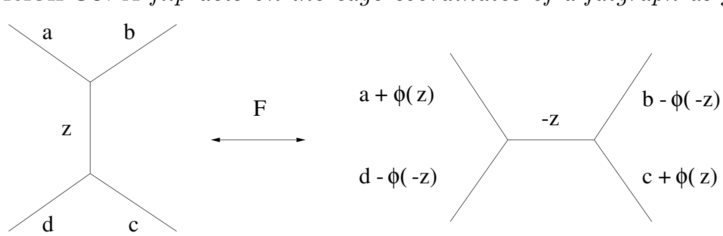
11.4 The mapping class group action on the Teichmüller spaces

In order to understand the action on $\mathcal{T}_g^s$ we to consider also $\mathcal{T}_{g,s;or}$.

The action of $\mathcal{M}_g^s$ on the Teichmüller space extends to an action of P_g^s to $\mathcal{T}_g^s$. Geometrically we can see it as follows. An element of $\mathcal{T}_g^s$ is a marked hyperbolic surface S. The marking comes from an ideal triangulation. If we change the triangulation by a flip, and keep the hyperbolic metric we obtain another element of $\mathcal{T}_g^s$.

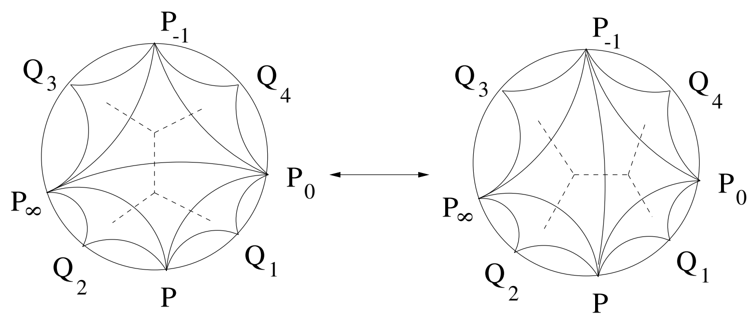
In the same way the $\mathcal{M}_{g,s}$ action on the Teichmüller space $\mathcal{T}_{g,s;or}$ extends to an action of the Ptolemy groupoid $P_{g,s}$. This action is very easy to understand in terms of coordinates. In more specific terms a flip between the graphs Γ and Γ' induces an analytic isomorphism $\mathbb{R}^{E_\Gamma} \to \mathbb{R}^{E_{\Gamma'}}$ by intertwining the coordinate systems t_Γ and $t_{\Gamma'}$. It is more convenient to identify $\mathbb{R}^{E_\Gamma}$ with a fixed Euclidean space, which is done by choosing a labelling $\sigma : E_\Gamma \to \{1, 2, ...\sharp E_\Gamma\}$ of its edges. Thus we have homeomorphism $\mathbf{t}_{\Gamma,\sigma} : \mathcal{T}_{g,s;or} \to \mathbb{R}^{\sharp E_\Gamma}$ given by $(\mathbf{t}_{\Gamma,\sigma}(S))_k = (\mathbf{t}_\Gamma(S))_{\sigma^{-1}(k)\in E_\Gamma}$. Further we can compare the coordinates $\mathbf{t}_{\Gamma,\sigma}$ and $\mathbf{t}_{F(\Gamma,\sigma)}$, for two labelled fatgraphs which are related by a flip. We can state:

Proposition 36. *A flip acts on the edge coordinates of a fatgraph as follows:*



where $\phi(z) = \log(1 + e^z)$. Here it is understood that the coordinates associated to the edges not appearing in the picture remain unchanged.

Proof. The flip on the graph corresponds to the following flip of ideal triangulations:



Then the coordinates a, b, c, d, z using the left-hand-side graph are the following cross-ratios: $a = [Q_3, P_\infty, P_0, P_{-1}]$, $b = [Q_4, P_{-1} P_\infty, P_0]$, $c = [Q_1, P_0, P_\infty, P]$, $d = [Q_2, P, P_0, P_\infty]$, $z = [P_{-1}, P_\infty, P, P_0]$. Let a', b', c', d', z' be the coordinates associated to the respective edges from the right-hand-side graph, which can again be expressed as cross-ratios as follows: $a' = [Q_3, P_\infty, P, P_{-1}]$, $b' = [Q_4, P_{-1}, P, P_0]$, $c = [Q_1, P_0, P_{-1}, P]$, $d = [Q_2, P, P_{-1}, P_\infty]$, $z = [P_\infty, P, P_0, P_{-1}]$. One uses for simplifying computations the half-plane model where, up to a Möbius transformation, the points P_{-1}, P_∞, P, P_0 are sent respectively into $-1, \infty, e^z$ and 0. The flip formulas follow immediately.

Remark 49. Similar computations hold for Penner's λ-coordinates on the decorated Teichmüller spaces. However the transformations of $\mathbf{R}^{6g-6+2s}$ obtained using λ-coordinates are *rational* functions.

Let us denote by $Aut^\omega(\mathbb{R}^m)$ the group of real analytic automorphisms of $\mathbb{R}^m$.

Corollary 19. *1. We have a faithful representation $\rho : \mathcal{M}_{g,s} \to Aut^\omega(\mathbb{R}^{6g-6+3s})$ induced by the $P_{g,s}$ action on the Teichmüller space $\mathcal{T}_{g,s;or}$ if $(g, s) \neq (1, 1)$.*
2. The groupoid $P_g^s \subset P_{g,s}$ leaves invariant the Teichmüller subspace $\mathcal{T}_g^s \subset \mathcal{T}_{g,s;or}$. Therefore the formula given in proposition 36 above for the flip actually yields a representation of P_g^s into $Aut^\omega(\mathbb{R}^{6g-6+2s})$. The restriction to the mapping class groups is a faithful representation $\rho : \mathcal{M}_g^s \to Aut^\omega(\mathbb{R}^{6g-6+2s})$ if $(g, s) \neq (1, 1)$, and a faithful representation of $\mathrm{PSL}(2, \mathbb{R})$ *when $(g, s) = (1, 1)$.*

Proof. The representation of $\mathcal{M}_{g,s}$ (respectively $\mathcal{M}_g^s$) is injective because the mapping class group acts effectively on the Teichmüller space. Therefore if the class of any (marked) Riemann surface is preserved by a homeomorphism then this homeomorphism is isotopic to the identity.

The invariance of the subspace $\mathcal{T}_g^s \subset \mathcal{T}_{g,s;or}$ by flips is geometrically obvious, but we write it down algebraically for further use. This amounts to check that the linear equations $t_\gamma = 0$, for $\gamma \in F_\Gamma$ are preserved. Let γ be a left-hand-turn path, which intersects the part of the graph shown in the picture, say along the edges labeled a, z, b. Then the flip of γ intersects the new graph along the edges labeled by $a + \phi(z)$ and $b - \phi(-z)$. The claim follows from the equality $z = \phi(z) - \phi(-z)$. The remaining three cases reduces to the same equation.

Remark 50. There is a Pt_g^s-action on the Teichmüller space but it is not free, and actually factors through P_g^s.

Remark 51. Assume that there exists an element $\mathbf{r} \in \mathcal{T}_g^s$, which is fixed by *some* $\psi \in \mathcal{M}_g^s$, i.e. $\varphi(\psi)(\mathbf{r}) = \mathbf{r}$. Then $\mathbf{r}$ is contained in some codimension two analytic submanifold $Q_g^s \subset \mathcal{T}_g^s$, and for a given $\mathbf{r}$ its isotropy group is finite. This is a reformulation of the fact that $\mathcal{M}_g^s$ acts properly discontinuously on the Teichmüller space with finite isotropy groups corresponding to the Riemann surfaces with non-trivial automorphism groups (biholomorphic). Moreover the locus of Riemann surfaces with automorphisms is a proper complex subvariety of the Teichmüller space, corresponding to the singular locus of the moduli space of curves.

11.5 Deformations of the mapping class group representations

We want to consider deformations of the tautological representation $\rho = \rho_0$ of $\mathcal{M}_g^s$ obtained in the previous section. We first restrict ourselves to deformations $\rho_h : \mathcal{M}_g^s \to Aut^\omega(\mathbb{R}^{6g-6+2s})$ satisfying the following requirements:

1. The deformation ρ_h extends to the Ptolemy groupoid P_g^s. In particular ρ_h is completely determined by $\rho_h(F)$ and $\rho_h(\tau_{(ij)})$.
2. The image of a permutation $\rho_h(\tau_{(ij)})$ is the automorphism of $\mathbb{R}^{6g-6+2s}$ given by the permutation matrix $P_{(ij)}$, which exchanges the i-th and j-th coordinates.
3. The image $F_h = \rho_h(F)$ of a flip has the same form as for $\rho_0(F)$, namely that given in the picture from proposition 36, but with a deformed function $\phi = \phi_h$, with $\phi_0 = \log(1 + e^z)$.
4. The linear subspace $\mathcal{T}_g^s \subset \mathcal{T}_{g,s;or}$ is invariant by ρ_h.

Proposition 37. *The real function $\phi : \mathbb{R} \to \mathbb{R}$ yield a deformation of the mapping class groups (respectively the Ptolemy groupoids) if and only if it satisfies the following functional equations:*

$$\phi(x) = \phi(-x) + x. \tag{11.1}$$

$$\phi(x + \phi(y)) = \phi(x + y - \phi(x)) + \phi(x). \tag{11.2}$$

$$\phi\left(\phi\left(x + \phi(y)\right) - y\right) = \phi(-y) + \phi(x). \tag{11.3}$$

Proof. The first equation is equivalent to the invariance of the linear equations defining the cusps. The other two equations follow from the cumbersome but straightforward computation of terms involved in the pentagon equation.

11.6 Belyi Surfaces

Let S be a compact Riemann surface. It is well known that there exists a non-constant meromorphic function on S, $\phi : S \to \mathbb{CP}^1$.

Definition 31. *The Riemann surface S is a* Belyi surface *if there exists a ramified covering $\phi : S \to \mathbb{CP}^1$, branched over 0, 1 and ∞.*

A surprising theorem of Belyi ([1]) states that:

Theorem 32. *S is a Belyi surface if and only if it is defined over $\overline{\mathbb{Q}}$ i.e. as a curve in $\mathbb{CP}^2$ its minimal polynomial lies over some number field.*

Following [22, 20] we can characterize Belyi surfaces in terms of fat graphs as follows:

Theorem 33. *A Riemann surface S can be constructed as $S(\Gamma) = S(\Gamma, \mathbf{0})$ for some trivalent fatgraph Γ if and only if S is a Belyi surface.*

Proof. We prove first:

Lemma 50. *Let $G \subset \mathrm{PSL}(2, \mathbb{Z})$ be a finite index torsion-free subgroup. Then $\mathbb{H}^2/G = S(\Gamma)$ for some trivalent fatgraph Γ.*

Proof. Remark that $A = \{z \in \mathbb{H}^2; 0 < \Re(z) < 1, |z| > 1, |z-1| > 1\}$, is a fundamental domain for $\mathrm{PSL}(2, \mathbb{Z})$, with the property that three copies of it around $\omega + 1$ fit together to give the ideal marked triangle. These three copies are equivalent by means of an order three elliptic element γ of $\mathrm{PSL}(2, \mathbb{Z})$.

A fundamental domain for G is composed of copies of A, and since G is torsion free the three copies A, $\gamma(A)$ and $\gamma^2(A)$ are not equivalent under G, thus they can all be included in the fundamental domain for G. In particular it exists a fundamental domain B for G which is made of copies of the ideal triangle I and hence it is naturally triangulated. Consider the graph Γ dual to this triangulation, which takes into account the boundary pairings, and which inherits an orientation from $\mathbb{H}^2/G$. Then $\mathbb{H}^2/G = S(\Gamma)$.

Lemma 51. *S is a Belyi surface if and only if we can find finitely many points on S, $\{p_1, \ldots, p_k\}$, such that $S - \{p_1, \ldots, p_k\}$ is isomorphic to $\mathbb{H}^2/G$, where G is a finite index torsion free subgroup of $\mathrm{PSL}(2, \mathbb{Z})$.*

Proof. Set $\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2 \right\}$. Then $F = \{z \in \mathbb{H}^2; 0 < \Re(z) < 2, |z - 1/2| > \frac{1}{2}, |z - 3/2| > \frac{1}{2}\}$ is a fundamental domain for $\Gamma(2)$ composed of 2 ideal triangles glued along a common edge. Thus the 3-punctured sphere $\mathbb{CP}^1 - \{0, 1, \infty\}$ is $\mathbb{H}^2/\Gamma(2)$. Moreover each ideal triangle is composed of three copies of the fundamental domain of $\mathrm{PSL}(2, \mathbb{Z})$. Therefore, the 3-punctured sphere is a six-fold branched covering of $\mathbb{H}^2/\mathrm{PSL}(2, \mathbb{Z})$.

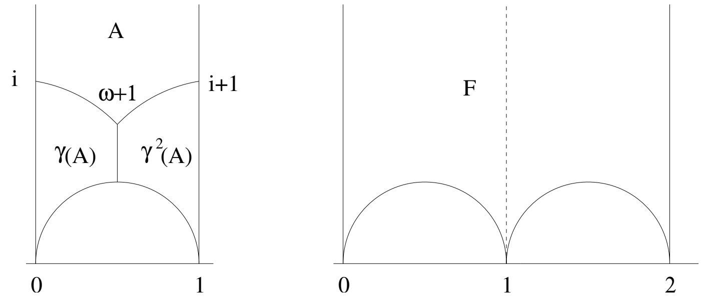


Fig. 11.3. Fundamental domains for $\mathrm{PSL}(2,\mathbb{Z})$ and $\Gamma(2)$

Let S be a compact Riemann surface. If S is a Belyi surface then $S - \{p_1, \dots, p_k\}$ is a regular smooth finite degree covering of $\mathbb{H}^2/\Gamma(2)$ and thus $S - \{p_1, \dots, p_k\} = \mathbb{H}^2/G$, where G is a finite-index subgroup of $\Gamma(2)$ (and hence of $\mathrm{PSL}(2,\mathbb{Z})$).

Conversely, if $S - \{p_1, \dots, p_k\} = \mathbb{H}^2/G$, where G is a finite index torsion free subgroup of $\mathrm{PSL}(2,\mathbb{Z})$, then $S - \{p_1, \dots, p_k\}$ is a finite-degree branched covering of $\mathbb{H}^2/\mathrm{PSL}(2,\mathbb{Z})$, which is a sphere with one cusp and 2 ramification points. Therefore, if we remove the 2 ramification points and their pre-images, we get that $S - \{p_1, \dots, p_k, \dots, p_n\}$ is a regular smooth finite-degree covering of the 3-punctured sphere, i.e. a Belyi surface.

These lemmas show that any Belyi surface can be constructed out of some fatgraph.

Conversely the fundamental polygon constructed for $G(\Gamma)$ is composed of copies of the ideal triangle. By decomposing each ideal triangle into three copies of the fundamental domain for $\mathrm{PSL}(2,\mathbb{Z})$, we see that $G(\Gamma)$ can be embedded as a finite-index torsion free subgroup of $\mathrm{PSL}(2,\mathbb{Z})$.

11.7 The geometry of the Teichmüller space

11.7.1 Symplectic structures for the Teichmüller space of punctured surfaces

The Teichmüller space $\mathcal{T}_g^s$ has a natural structure of complex manifold. Let us recall some of its features. Suppose that the Riemann surface S is uniformized by the Fuchsian group $G \subset \mathrm{PSL}(2,\mathbb{R})$.

One considers first the vector space $Q(S) = Q(G)$ of integrable holomorphic quadratic differentials on S. An element $\varphi \in Q(S)$ is a holomorphic function $\varphi(z)$ on $\mathbb{H}^2$ satisfying $\varphi(\gamma(z))\gamma'(z)^2 = \varphi(z)$ for all $\gamma \in G$, and $\int_F |\varphi|$ is finite, where F is a fundamental domain for G. Then φ induces a symmetric tensor of type $(2,0)$ on S.

Let then $M(S)$ be the space of G-invariant Beltrami differentials. These are measurable, essentially bounded functions $\mu : \mathbb{H}^2 \to \mathbb{C}$ satisfying $\mu(\gamma(z))\frac{\overline{\gamma'(z)}}{\gamma'(z)} = \mu(z)$ for all $\gamma \in G$, and hence define a $(-1,1)$ tensor on S.

There is a natural pairing $(,) : M(S) \times Q(S) \to \mathbb{C}$ given by $(\mu, \varphi) = \int_F \mu\varphi$, with null space $N(S) \subset M(S)$ which induces a duality isomorphism between $M(S)/N(S)$ and $Q(S)$.

The holomorphic cotangent space at the point $[S] \in \mathcal{T}_g^s$ is identified with $Q(S)$ and thus the tangent space is naturally isomorphic to $M(S)/N(S)$. Weil introduced a hermitian product on $Q(S)$ defined in terms of the Petersson product for automorphic forms. This yields the Weil-Petersson (co)metric on $\mathcal{T}_g^s$:

$$< \varphi, \psi >= \frac{1}{2}\mathsf{Re}\int_{\mathbb{H}^2/G} \varphi\overline{\psi}\,(\mathsf{Im}z)^{-2}\,, \quad \text{for } \varphi, \psi \in Q(S).$$

Remark 52. The Weil-Petersson metric is Kähler, it has negative holomorphic sectional curvature and is invariant under the action of the mapping class group.

The Kähler form of the Weil-Petersson metric is a symplectic form ω_{WP}. In the case of closed surfaces Wolpert ([32]) derived a convenient expression for ω_{WP} in terms of Fenchel-Nielsen coordinates:

$$\omega_{WP} = -\sum_j d\,\tau_j \wedge d\,l_j.$$

Recall that a pair of pants $\Sigma_{0,3}$ has a hyperbolic structure with geodesic boundary. The lengths $l_j \in \mathbb{R}_+$ of the boundary circles can be arbitrarily prescribed. To each decomposition of S into pairs of pants $P_1, ..., P_{2g-2}$ we have therefore associated the lengths of their boundary geodesics $l_1, ..., l_{3g-3}$. In fact given pairs of pants, not necessarily distinct, P_1 and P_2 with boundary circles c_1 on P_1 and c_2 on P_2, of the same length we can glue the pants by identifying c_1 with c_2 by an isometry. The hyperbolic metric extends over the connected sum. Therefore we can glue together the pants $P_1, ..., P_{2g-2}$ to obtain the Riemann surface S. If a length $l = 0$ then this corresponds to the situation where the surface has a cusps. We can therefore extend this description to punctured surfaces Σ_g^n with cusps at punctures. The pants decomposition is specified therefore by $3g-3+n$ geodesics on S. Each boundary circle c belongs to two pairs of pants P_j and P_k. The geodesics joining the circles of P_j to the circles of P_k intersect c into two points. The parameter τ_j is the (signed) hyperbolic distance between these two points. The parameters (τ_j, l_j) are the Fenchel Nielsen coordinates on $\mathcal{T}_g^s$.

Fricke and Klein established that, if one carefully choose the curves $\gamma_1, ..., \gamma_{6g-6+2n}$ then the associated lengths l_j can also give local coordinates on $\mathcal{T}_g^s$. A typical example is to pick up first the curves $\gamma_1, ..., \gamma_{3g-3+n}$ arising from a pants decomposition, and then a dual pants decomposition obtained as follows. Consider the pieces of geodesics which yield the canonical points on the circles, and then identify combinatorially the canonical points. We obtained this way a family of closed loops $\gamma_{3g-3+n+1}, ..., \gamma_{6g-6+2n}$. Wolpert ([33],Lemma 4.2, 4.5) expressed the Kähler form in these coordinates:

Lemma 52. *Assume that $l_1, ..., l_{6g-6+2n}$ provide local coordinates on $\mathcal{T}_g^s$ and denote:*

$$\alpha^{jk} = \sum_{p \in \gamma_j \cap \gamma_k} \cos \theta_p,$$

where θ_p is the angle between the geodesic γ_j and γ_k at the point p. Let $W = (w_{jk})_{j,k}$ be the inverse of the matrix $A = (\alpha^{jk})_{j,k}$. Then the Weil-Petersson form is:

$$\omega_{WP} = -\sum_{j<k} w_{jk}\, d\, l_j \wedge d\, l_k.$$

11.7.2 Poisson structure for the Teichmüller space of surfaces with boundary

Let G be a connected Lie group, which will be most of the time $\mathrm{PSL}(2, \mathbb{R})$ in this section. Set $M(\Sigma, G) = \mathrm{Hom}(\pi_1(\Sigma), G)/G$ for the moduli space of representations of the fundamental groups.

Goldman ([8]) proved that $M(\Sigma, G)$ is endowed with a natural symplectic structure, whenever Σ is a closed oriented surface. Moreover Fock and Rosly ([6]) was able to show more generally that there is a Poisson structure on $M(\Sigma, G)$, even in the case when Σ is a surface with boundary. Furthermore the symplectic leaves of this structures are precisely the singular submanifolds $M(\Sigma, G)_{\lambda_1, ..., \lambda_s}$, where λ_j is the conjugacy class of the holonomy around the j-th boundary component.

Notice also that Zocca have shown that $M(\Sigma, G)$ has a pre-symplectic structure, whose restriction to the symplectic leaves is the symplectic form.

11.7.3 Penner's decorated Teichmüller space

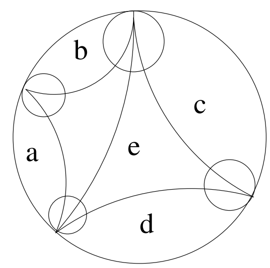
Penner ([22, 23] considered the space $\widetilde{\mathcal{T}}_g^s$ of cusped Riemann surfaces endowed with a horocycle around each puncture, and called it the decorated Teichmüller space. There is a natural family of coordinates (called lambda lengths), associated to the edges of an ideal triangulation $\Delta = \Delta(\Gamma)$ of the surface. For each such edge e one puts $\lambda_e = \sqrt{2exp(\delta)}$, where δ is the signed hyperbolic distance between the two horocycles centered at the two endpoints of the edge e. The sign convention is that $\lambda_e > 0$ if the horocycles are disjoint. It is not difficult to see that these coordinates give a homeomorphism $\widetilde{\mathcal{T}}_g^s \to \mathbb{R}^{6g-6+3s}$. The map which forgets the horocycles $\pi : \widetilde{\mathcal{T}}_g^s \to \mathcal{T}_g^s$ is a fibration having $\mathbb{R}_+^s$ as fibers. Moreover:

Lemma 53. *The projection π is expressed in terms of Penner and Fock coordinates as follows:*

$$\pi\left((\lambda_e)_{e\in\Delta(\Gamma)}\right) = \left(\log\frac{\lambda_a\lambda_c}{\lambda_b\lambda_d}\right)_{e\in\Delta(\Gamma)},$$

where, for each edge e we considered the quadrilateral of edges a, b, c, d, uniquely determined by the following properties:

- *the cyclic order a, b, c, d is consistent with the orientation of Σ_g^s.*
- *e is the diagonal separating a, b from c, d (see the figure 53).*
- *each triangle of Δ has an orientation inherited from Σ_g^s, in particular the edge e is naturally oriented. We ask that a (and d) be adjacent to the startpoint of e, while b and c is adjacent to the endpoint of e.*



Proof. The proof is a mere calculation.

Proposition 38. *The pull-back $\pi^*\omega_{WP}$ of the Weil-Petersson form on the decorated Teichmüller space $\widetilde{\mathcal{T}}_g^s$ is given in Penner's coordinates as:*

$$\pi^*\omega_{WP} = -2\sum_{T\subset\Delta} d\log\lambda_a\wedge d\log\lambda_b + d\log\lambda_b\wedge d\log\lambda_c + d\log\lambda_c\wedge d\log\lambda_a,$$

where the sum is over all triangles T in Δ whose edges have lambda lengths a, b, c in the cyclic order determined by the orientation of Σ_g^s.

Proof. See [24], Appendix A.

Remark 53. For dimensional reasons the pre-symplectic form $\pi^*\omega_{WP}$ is degenerate.

Proposition 39. *The Poisson structure on $\mathcal{T}_{g,s;or}$ is given by the following formula in the Fock coordinates (t_e):*

$$P_{WP} = \sum_{T\subset\Delta} dt_a\wedge dt_b + dt_b\wedge dt_c + dt_c\wedge dt_a,$$

where the sum is over all triangles T in Δ whose edges are a, b, c in the cyclic order determined by the orientation of Σ_g^s. This Poisson structure is degenerate. Moreover $\mathcal{T}_g^s\subset\mathcal{T}_{g,s;or}$ is a symplectic leaf and hence the restriction of P_{WP} is the Poisson structure dual to the Weil-Petersson symplectic form ω_{WP}.

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12

Riemann surfaces

12.1 Generalities

Note that Radó proved in 1925 that all Riemann surfaces are second-countable. This cannot be extended to higher dimensions, as Prüfer's complex surface is not second-countable.