Hyperbolic plane $\mathbb{H}^2$ $\qquad \frac{|dz|}{\operatorname{Im} z}$ $\qquad z \mapsto z+t$, $\; z \mapsto -1/z$

$$T^1(\mathbb{H}^2) \equiv PSL_2(\mathbb{R}) \qquad \mathbb{H}^2 = SO(2)\backslash PSL_2(\mathbb{R})$$

pf action transitive $\qquad \begin{pmatrix} (\operatorname{Im} w)^{1/2} & \operatorname{Re} w \\ 0 & (\operatorname{Im} w)^{-1/2} \end{pmatrix} \qquad i \mapsto w$

stabiliser of i

Lemme $\qquad \operatorname{Im} \frac{az+b}{cz+d} = \det(\ ) \frac{\operatorname{Im} z}{|cz+d|^2}$

$$\frac{d}{dz}\left(\quad \right) = \det(\ ) \frac{1}{(cz+d)^2}$$

Ideal boundary $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\} \equiv \mathbb{P}(\mathbb{R}^2)$

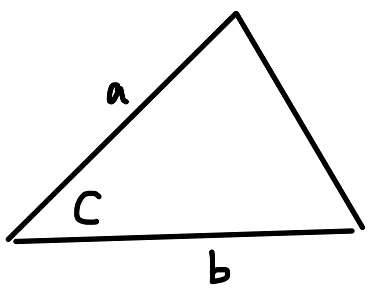
Lemme $PGL_2(\mathbb{R})$ action simply transitive on distinct triples $(\partial\mathbb{H})^3$

pf $\qquad a \to 0$, $b \to \infty$, $c \to 1$ $\qquad z \mapsto \frac{z-a}{z-b} \cdot \frac{c-b}{c-a}$ $\quad$ must be >0 to get in $SL_2(\mathbb{R})$

geodesics $\qquad$ ideal triangles

horocycles

Lemme $\qquad \cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C$



$$c^2 = a^2 + b^2 - 2ab \cos C$$

$SL_2(\mathbb{Z}) \simeq PSL_2(\mathbb{Z}) \times$

$SL_2(\mathbb{R})$ is generated by $\left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, s \in \mathbb{R} \right\}$ & $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

translation $z \mapsto z+s$

inversion $z \mapsto -1/z = -\bar{z}/|z|^2$

$SL_2(\mathbb{Z})$ generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ & $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

fundamental domain for $(P)SL_2(\mathbb{Z}) \curvearrowright \mathbb{H}^2 \subset \{|\mathrm{Re}\, z| \leq \frac{1}{2}\} \cap \{|z| \geq 1\}$

ie f dom for $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^s$ $\langle z \mapsto z+s \rangle$

for $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\langle z \mapsto -1/z \rangle$

proof $\exists w \in \Gamma z$ st $-\frac{1}{2} \leq \mathrm{Re}\, w \leq \frac{1}{2}$ (easy)

$|w| \geq 1$ then done

$|w| < 1$ then $|-1/w| = 1/|w| > 1$ $\square$

Discussion ideal triangles

Def Farey graph / complex

0-cells $= \begin{pmatrix} m \\ n \end{pmatrix} \in \mathbb{Z}^2$ primitive

1-cells $=$ pairs of 0 cells det $= \pm 1$

$=$ bases of $\mathbb{Z}^2$ as a $\mathbb{Z}$ module

2-cells $=$ triples / superbases

Def $\pm e_1, e_2, e_3$ primitive $e_1 + e_2 + e_3 = 0$

ex $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

geometric realisation in $\mathbb{H}^2 \cup \partial \mathbb{H}^2$

$\mathbb{P}(\mathbb{R}^2)$

Marked parallelograms & Tori

20.1 Parallelograms

Say that a *marked parallelogram* is a parallelogram P with a distinguished vertex v, a distinguished first side e_1, and a distinguished second side e_2. The sides e_1 and e_2 should meet at v, as in Figure 20.1. We say that two marked parallelograms P_1 and P_2 are *equivalent* if there is an orientation-preserving similarity, i.e., a translation followed by a dilation followed by a rotation, that maps P_1 to P_2 and preserves all the markings.

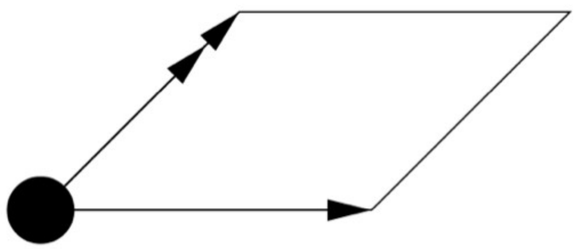


Figure 20.1. A marked parallelogram

We think of P as a subset of $\boldsymbol{C}$. If we have a marked parallelogram, we can translate it so that $v = 0$ and e_1 points from 0 to 1. Then e_2 points from 0 to some $z \in \boldsymbol{C} - \boldsymbol{R}$. We only consider "half" of the possibilities, the case when $z \in \boldsymbol{H}^2$, considered as the upper half plane of $\boldsymbol{C}$.

Exercise 1. Prove that $z(P_1) = z(P_2)$ if and only if P_1 and P_2 are equivalent.

Definition 20.2. A *marked flat torus* is a triple (Σ, T, ϕ), where T is a flat torus and $\phi : \Sigma \to T$ is an orientation-preserving homeomorphism. We say that two triples (Σ, T_1, ϕ_1) and (Σ, T_2, ϕ_1) are *equivalent* if there is an orientation-preserving similarity $f : T_1 \to T_2$ such that $f \circ \phi_1$ and ϕ_2 are homotopic maps.

Mapping class action

Given a triple (Σ, T, ϕ), we define the new triple $(\Sigma, T, \phi \circ g^{-1})$. That is, we keep the same surface T, but we change $\phi : \Sigma \to T$ to the map given by the composition $\Sigma \to \Sigma \to T$, with the first arrow given by g^{-1}. We use g^{-1} in place of the more obvious choice of g for technical reasons that we will explain momentarily.

Exercise 4. Prove that (Σ, T_1, ϕ_1) and (Σ, T_2, ϕ_2) are equivalent if and only if $(\Sigma, T_1, \phi_1 \circ g)$ and $(\Sigma, T_2, \phi_2 \circ g)$ are equivalent.

The group $\mathcal{G}$ *acts* on the space $\mathcal{T}$ in the sense that

$$g_1(g_2(x)) = (g_1 \circ g_2)(x), \tag{82}$$

for all $g_1, g_2 \in \mathcal{G}$ and all $x \in \mathcal{T}$. Here $g_1 \circ g_2$ means "first do g_2 and then do g_1". To see this, let x be a point represented by the triple (Σ, T, ϕ). We compute

$$\begin{aligned} g_1(g_2(x)) &= g_1(\Sigma, T, \phi \circ g_2^{-1})(\Sigma, T, \phi \circ g_2^{-1} \circ g_1^{-1}) \\ &= (\Sigma, T, \phi \circ (g_1 \circ g_2)^{-1}) = (g_1 \circ g_2)(x). \end{aligned}$$

of the parallelogram $(0, 1, z, 1 + z)$, and so that ϕ is induced by the linear transformation carrying $(1, 0)$ to $(1, 0)$ and $(0, 1)$ to (x, y). Here $z = x + iy$. When we lift ϕ to the universal covers of Σ and T, respectively, we get the same linear transformation. In other words, the linear transformation

$$\widehat{\phi} = \begin{bmatrix} 1 & x \\ 0 & y \end{bmatrix} \tag{84}$$

induces the homeomorphism ϕ. The linear transformation

$$\widehat{\phi} \circ g^{-1} = \begin{bmatrix} d - cx & -b + ax \\ -cy & ay \end{bmatrix} \tag{85}$$

induces the homeomorphism $\phi \circ g^{-1}$.

$$\begin{pmatrix} d - cx & -cy \\ cy & d - cx \end{pmatrix} \begin{pmatrix} & \\ & \end{pmatrix} = \begin{pmatrix} |cz + d|^2 & * \\ 0 & * \end{pmatrix}$$

up to scalar $\in SO(2)$