Geometry, Groups, Geodesics

Free group

$$\Gamma' \to \mathbb{Z} * \mathbb{Z} \to \mathbb{Z}^2$$

$$\begin{array}{c} Z(SL_2(\mathbb{Z})) \\ \downarrow \\ SL_2(\mathbb{Z}) \triangleleft GL_2(\mathbb{Z}) \simeq \mathrm{Aut}(\mathbb{Z}^2) \\ \downarrow \\ PSL_2(\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} \end{array}$$

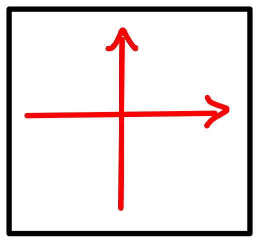
Fundamental group

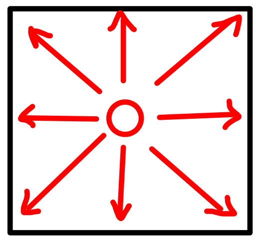
a group is free iff

π_1(bouquet of S^1)

acts freely on a tree = universal cover of bouquet of S^1

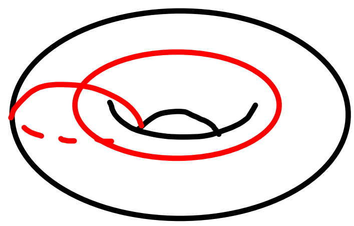
$$\pi_1(\mathbb{T}^2) \simeq \mathbb{Z}^2$$

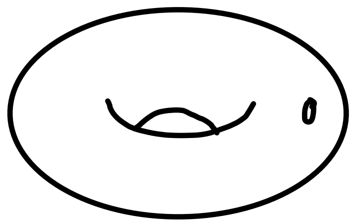




$$S^1 \vee S^1 \underset{\simeq}{\hookrightarrow} \mathbb{T}^2_x \overset{\iota}{\hookrightarrow} \mathbb{T}^2 \equiv S^1 \times S^1$$

$$\ker \hookrightarrow \pi_1(\mathbb{T}^2_x) \twoheadrightarrow \pi_1(\mathbb{T}^2)$$





Representation variety

G = Lie gp eg $SL_n(\mathbb{R})$

$\mathrm{Inn}(G)$
$\downarrow$
$\mathrm{Aut}(G)$ action $\qquad \mathrm{Hom}(\pi_1, G) \hookrightarrow G^n$, n = # gens of π_1
$\downarrow \qquad\qquad\qquad\qquad \downarrow$
$\mathrm{Out}(G)$ action $\qquad \mathcal{X}(\pi_1, G) \hookrightarrow G^n /\!/ G$ diagonal action by conjugation

Example lattices in $\mathbb{R}^2$ $\mathbb{E}^2$

$$\mathbb{R}^2 \hookrightarrow G = \mathrm{Isom}^+ \mathbb{R}^2 \to SO(2)$$

$T_v \quad x \mapsto x + \vec{v}$ $\qquad R_\theta \quad x \mapsto R_\theta x$

$$R_\theta \circ T_v \circ R_\theta^{-1}(x) = x + R_\theta \vec{v}$$

Quotient spaces

$$\Gamma = \langle v_1, v_2 \rangle, \quad |v_1 \wedge v_2| = 1$$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "R_\theta"] \arrow[d] & \mathbb{R}^2 \arrow[d] \\ \mathbb{R}^2/\Gamma \arrow[r] & \mathbb{R}^2/R_\theta\Gamma \end{tikzcd} \qquad \begin{tikzcd} \mathbb{E}^2 \arrow[r, "R_\theta", "\simeq"'] \arrow[d] & \mathbb{E}^2 \arrow[d] \\ \mathbb{E}^2/\Gamma \arrow[r] & \mathbb{E}^2/R_\theta\Gamma \end{tikzcd}$$

isometric

up to isometry $\mathbb{E}^2/\Gamma$
only depends on the orbit of Γ
under conjugation by G

$$\mathcal{L}(\mathbb{E}^2) \hookrightarrow G \times G /\!/ G$$

moduli = fns on the space

dimensions $3 + 3 - 3 = 3$

1/ $|v_1 \wedge v_2|$ = volume of $\mathbb{E}^2/\Gamma$

2/ $\mathrm{sys}(\ ) = \inf_{v \in \Gamma} \|v\|$

$\dfrac{\mathrm{sys}}{\sqrt{\mathrm{vol}}}$ = Hermite invariant

Lemme $\quad 0 < \dfrac{\mathrm{sys}}{\sqrt{\mathrm{vol}}} \leq 2/\sqrt{\pi} < 2/\sqrt{3} = (4/3)^{\frac{1}{2}}$

3/ in fact $\|v_1\|, \|v_2\|$, $v_1 v_2$ are parameters
$\|v_2 - v_1\|$

these are the side lengths of a Δ

Ex $\quad \mathbb{Z} * \mathbb{Z} \overset{\rho}{\hookrightarrow} SL_2(\mathbb{R})$

$$\begin{array}{l} \mathrm{Hom}(\qquad) \quad - \quad 6\ d \\ \downarrow \\ \chi(\qquad) \hookrightarrow \mathbb{R}^3 \\ (a, b, ab) \mapsto (\mathrm{tr}\, a, \mathrm{tr}\, b, \mathrm{tr}\, ab) \end{array}$$

Lemme $\quad A, B \in SL_2(\mathbb{R})$

$$\mathrm{tr}\, AB + \mathrm{tr}\, A^{-1}B = \mathrm{tr}\, A\, \mathrm{tr}\, B$$

$$\mathrm{tr}\, AB + \mathrm{tr}\, AB^{-1} = \mathrm{tr}\, A\, \mathrm{tr}\, B$$

CH $\quad A^2 - \mathrm{tr}\, A\, A + \det A\, I_2 = O_2$

$$AB - \mathrm{tr}\, A\, B + A^{-1}B = O_2$$

Cor $\quad \mathrm{tr}\, A^2 = (\mathrm{tr}\, A)^2 - 2$

$$\mathrm{tr}\, ABA^{-1}B^{-1} = (\mathrm{tr}\, A)^2 + \qquad\qquad -2$$

$= (ABA^{-1})B^{-1} - ABA^{-1}B$

$= \quad '' \quad - (AB)(A^{-1}B) + ABB^{-1}A$

$= \mathrm{tr}\, B^2 - \mathrm{tr}\, AB\, (\mathrm{tr}\, A + \mathrm{tr}\, B - \mathrm{tr}\, AB) + \mathrm{tr}\, A^2$

$$\begin{pmatrix} x & -t \\ 1/t & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & y \end{pmatrix} = \begin{pmatrix} t & * \\ * & 1/t \end{pmatrix} \qquad z = t + 1/t$$

$$t^2 - zt + 1 = 0$$

$$t \in \mathbb{R} \Rightarrow z^2 - 4 \geq 0$$

$\rho \in \mathrm{Hom}(\mathbb{Z} * \mathbb{Z}, SL(2, \mathbb{R})) \Rightarrow \chi(\rho) \notin \{\max(x, y, z) \leq 2\}$