Seeking flat tori — a diploid approach

Alba Málaga — with Samuel Lelièvre and Pierre Arnoux

July, 2022

Thanks

Thanks to IMPA, ICERM and CIRM for their help in the realisation of this work.

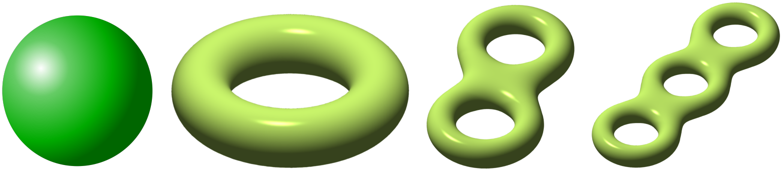
  

We thank our universities and the French university system for our permanent jobs.

Surfaces of constant curvature

- Any conformal compact surface can be endowed with a Riemannian metric of constant curvature.
- For a sphere, this metric is unique and has strictly positive curvature
- Is is easy to give an isometric model in $\mathbb{R}^3$
- For a surface of genus larger than 1, this metric has strictly negative curvature
- For a torus, this metric has curvature 0



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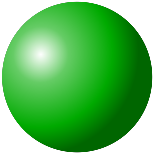

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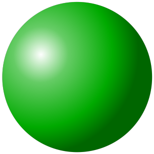

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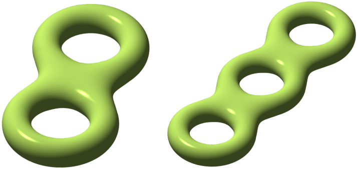

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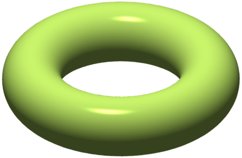
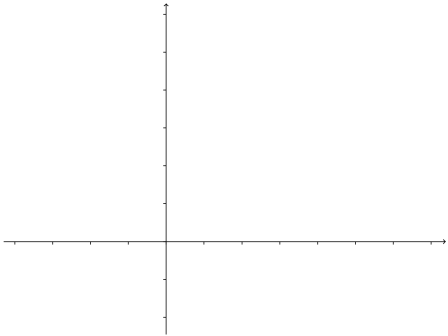


Flat tori

- The universal cover of a flat torus is $\mathbb{C}$
- The torus appears as the quotient $\mathbb{C}/\Lambda$
- Where Λ is a lattice in $\mathbb{C}$
- One can reconstruct the torus by gluing the boundary of a fundamental domain
- For example a parallelogram
- The space of lattices can be seen as the modular surface $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$.



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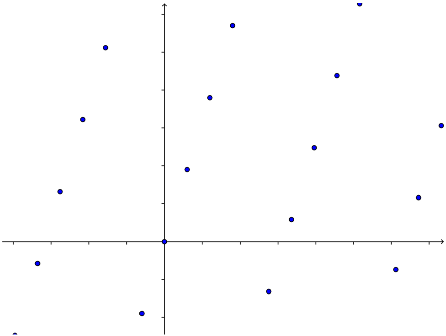

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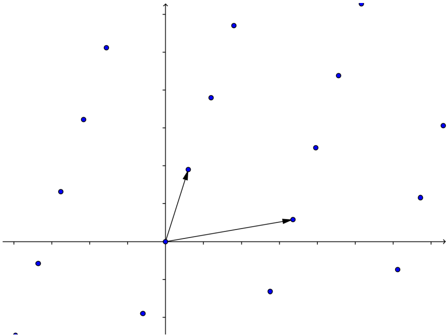

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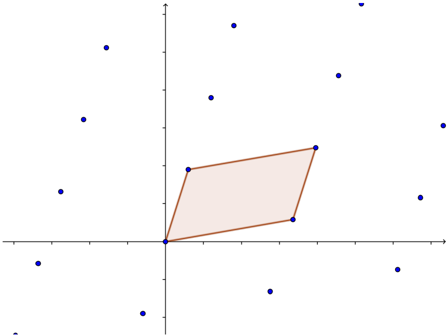

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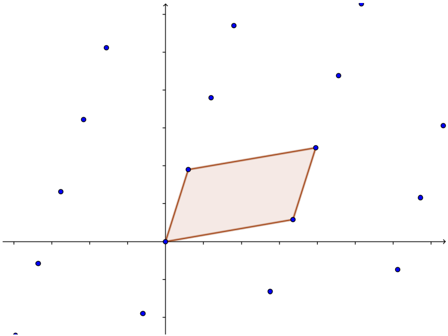

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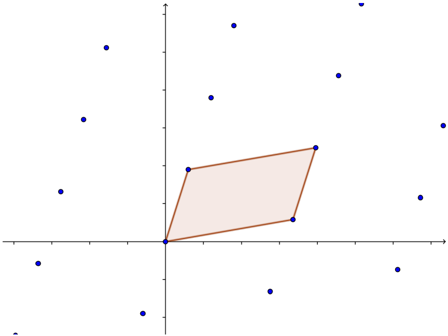
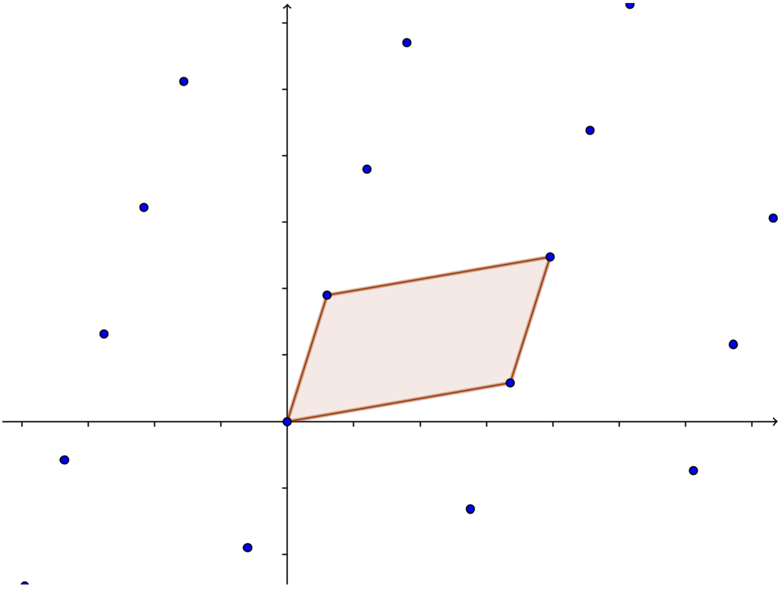


$SL(2,\mathbb{Z})\backslash\mathbb{H}$ as a modular curve of tori



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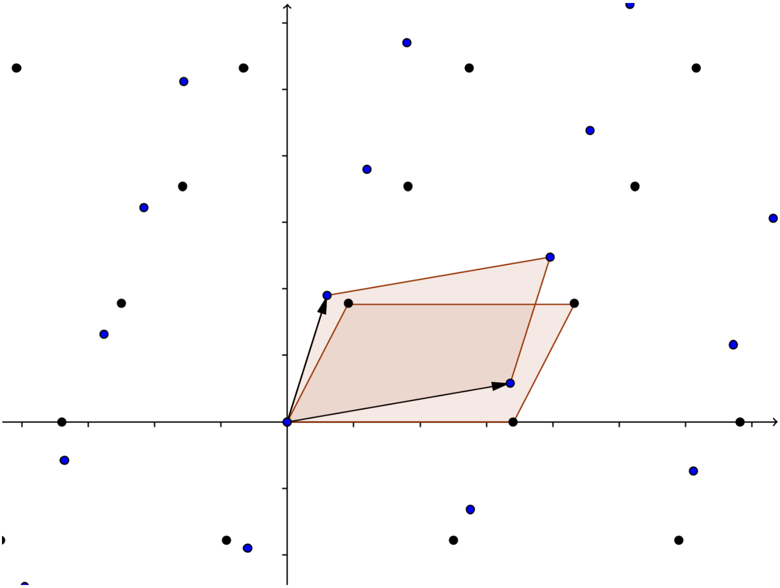

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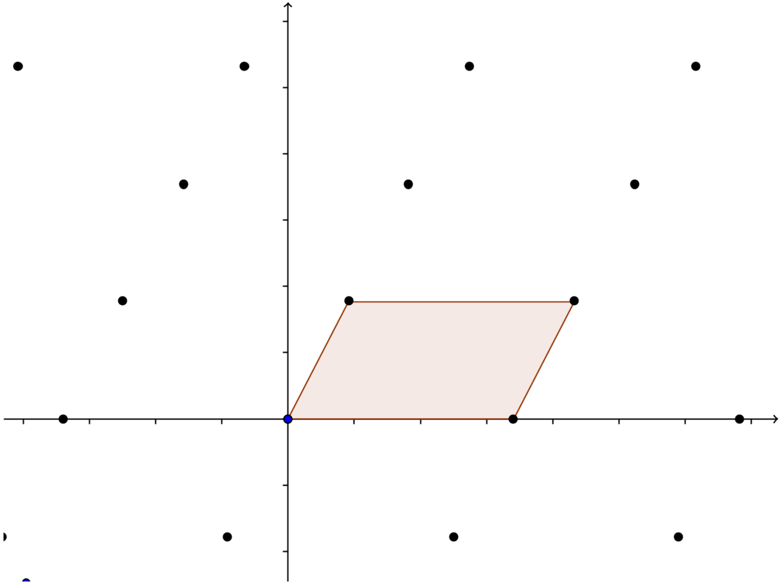

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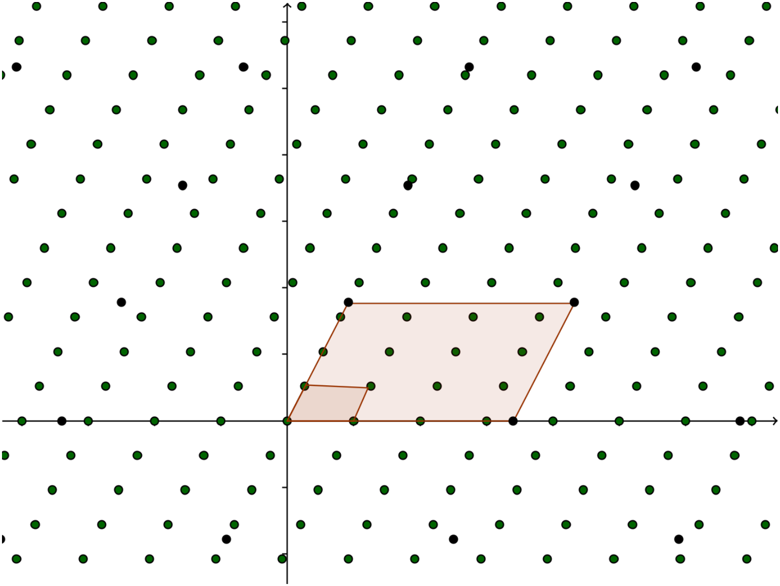

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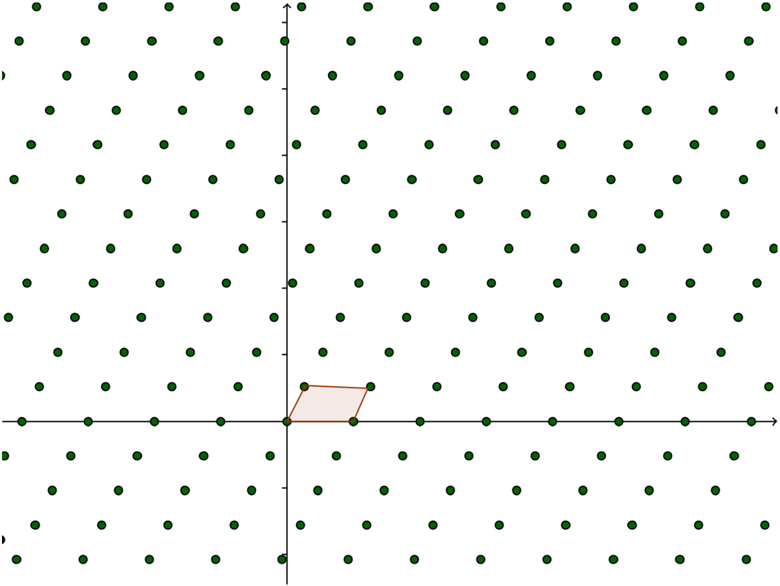

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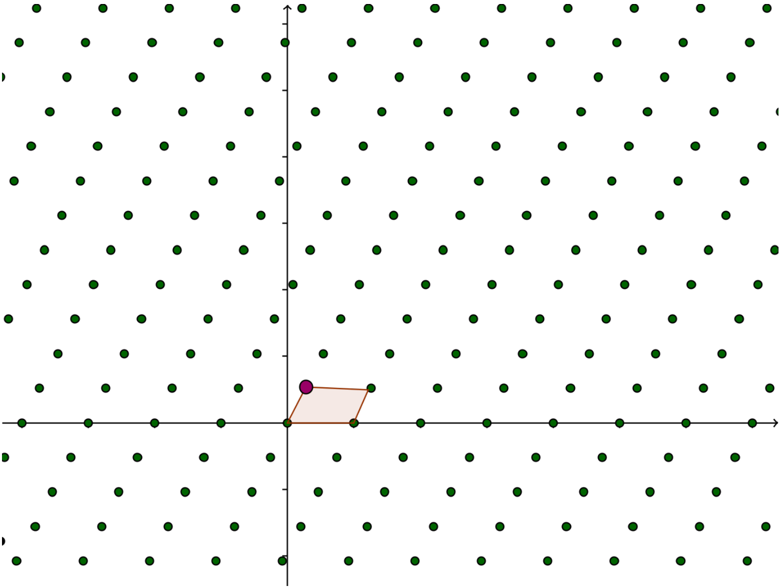

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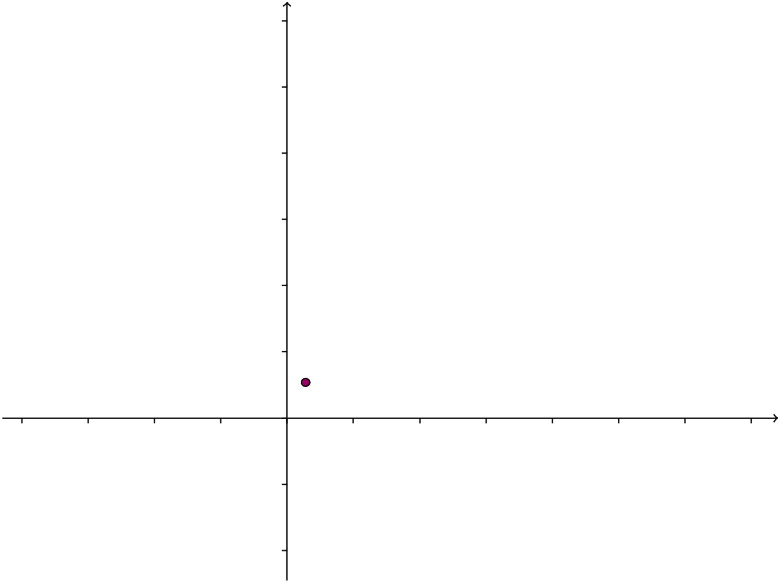

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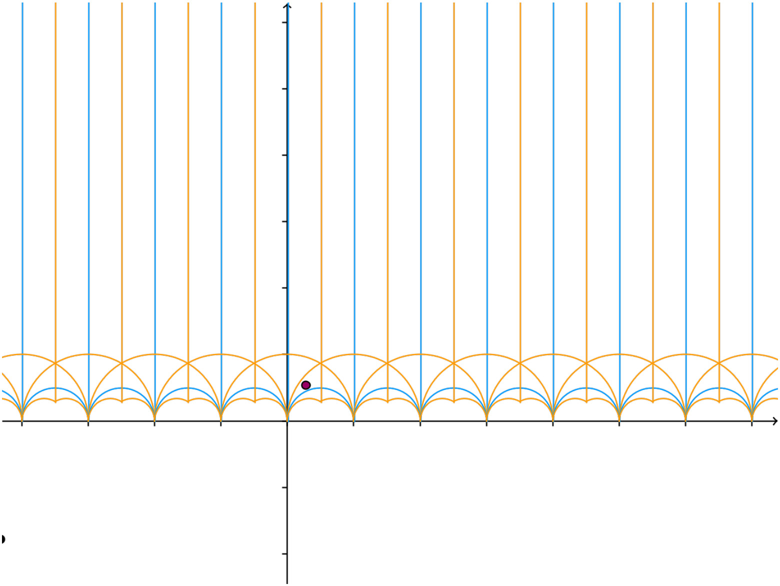

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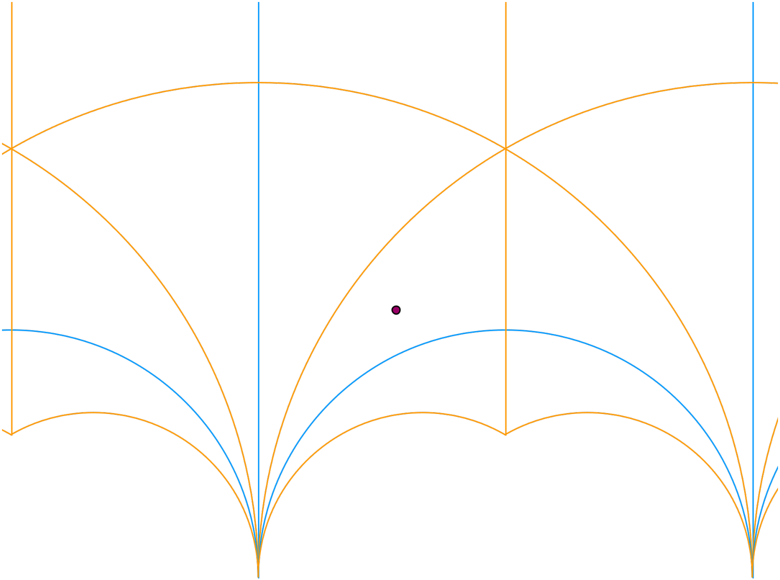
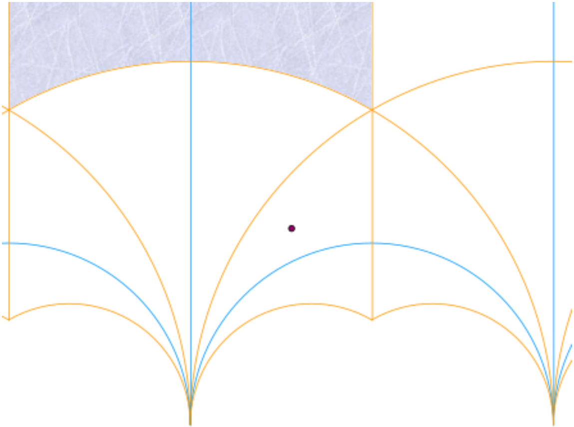


Fundamental domains for $SL(2, \mathbb{Z})$ acting on $\mathbb{H}$



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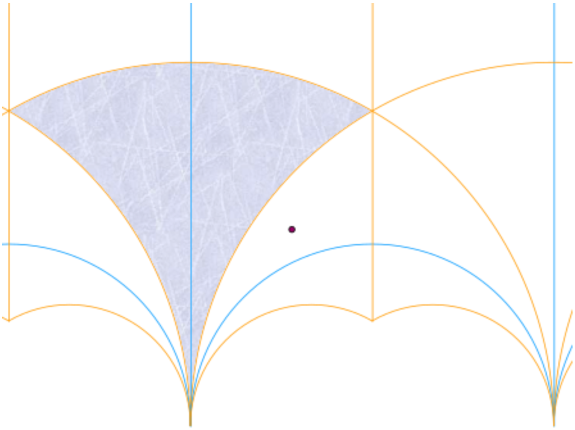

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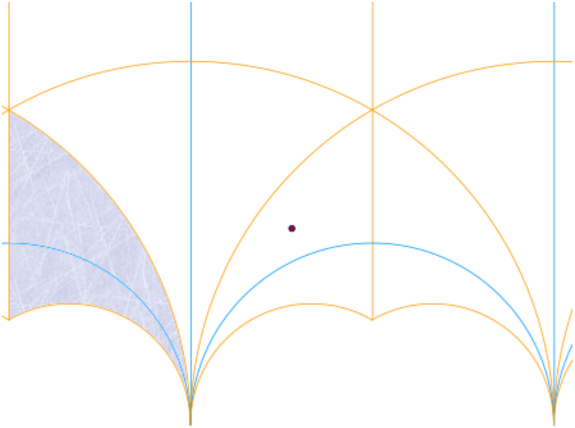

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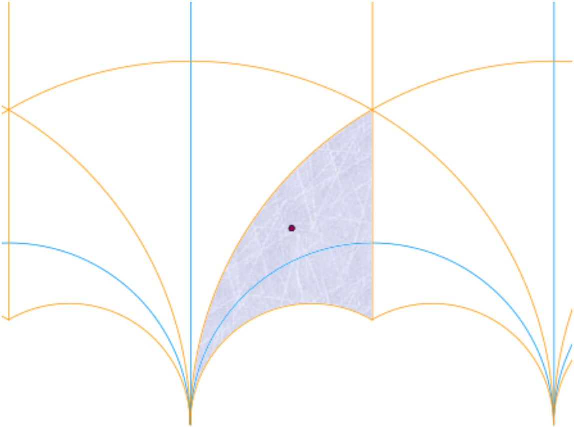

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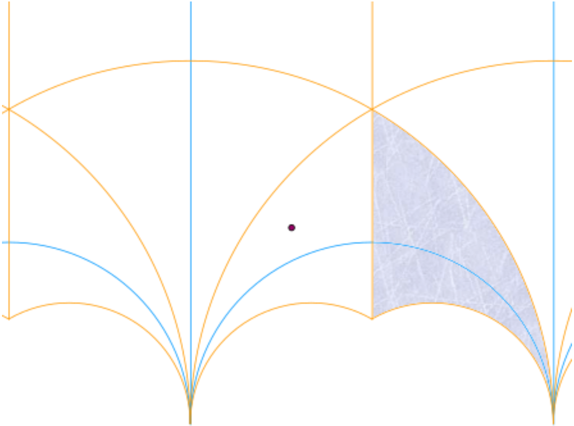

also see `https://p3d.in/MGPfJ`

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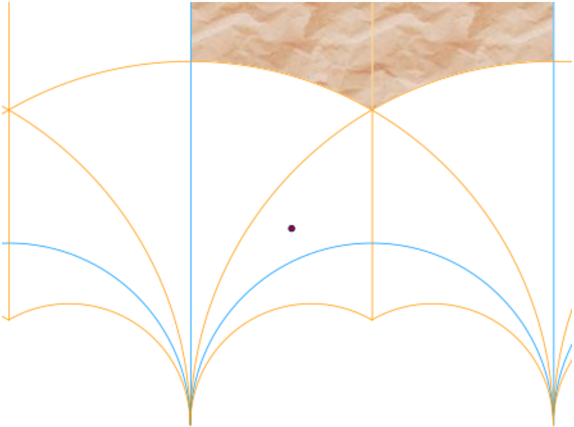

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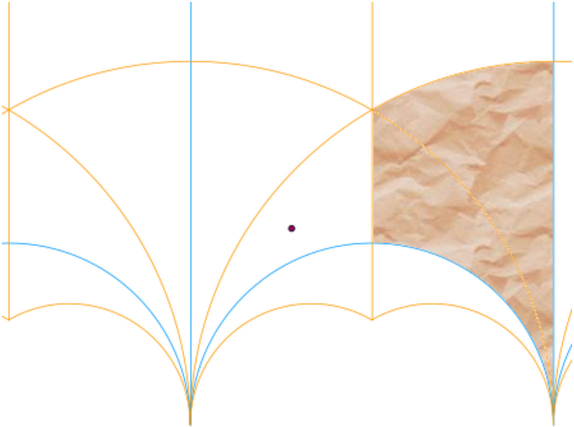

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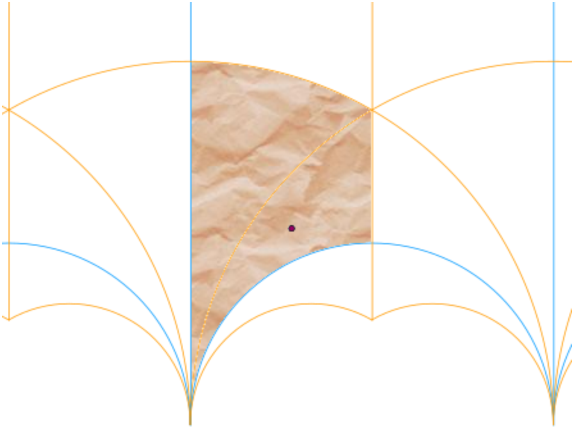

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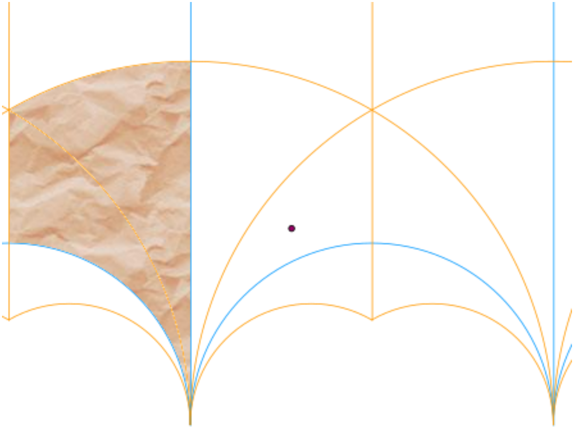

The question

- Easy: constant curvature sphere isometrically embedded in $\mathbb{R}^3$
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- Not if the embedding is C^2
- Every complete surface of curvature 0 in $\mathbb{R}^3$ is a cylinder
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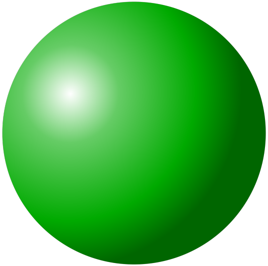

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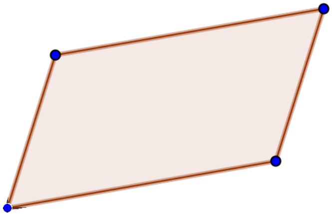

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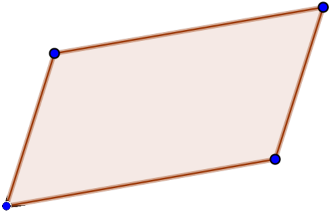

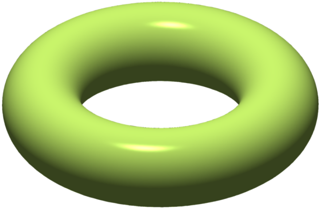

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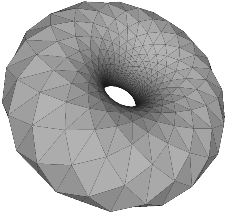
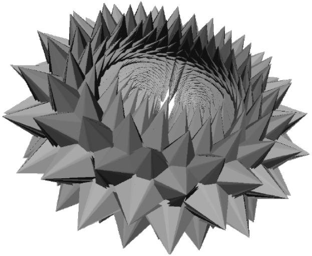
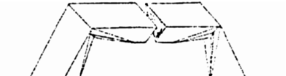
Another form of the question

- Is there a piecewise-linear isometric embedding of a flat torus?
- ... an origami embedding
- ... in which every vertex has cone angle 2π?
- This is in fact possible.
- Various answers:
  - Burago and Zalgaller's general construction (1996)
  - Zalgaller's "long tori" as bendings of long cylinders (2000)
  - Quintanar's finite corrugations of the square flat torus (2019)
  - Diplotori, an elementary construction (...-2021)





F. Tallerie



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- joining two parallel circles with a common vertical axis

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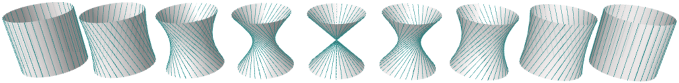

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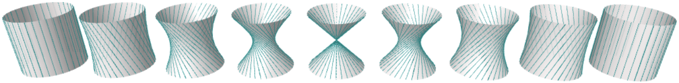

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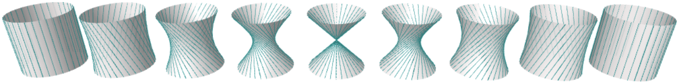

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PL-oids

- Consider a regular polygon with n vertices $P_0, \ldots, P_{n-1}$ centered at 0 in the plane $z = 0$
- and a regular polygon of the same size, with vertices $Q_0, \ldots, Q_{n-1}$ turned by an angle α, centered on the vertical axis in the plane $z = h$
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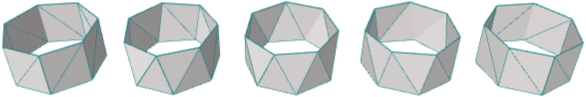

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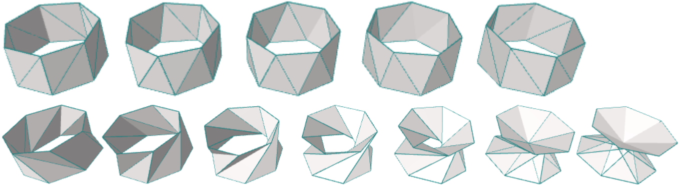

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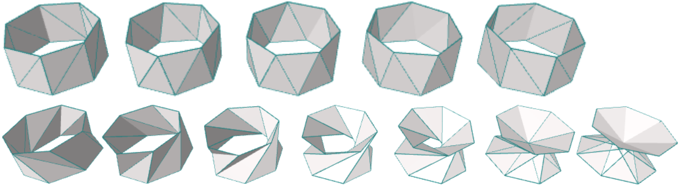

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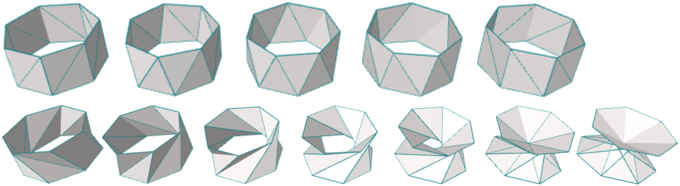

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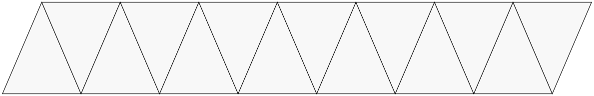

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PL-oids (2)

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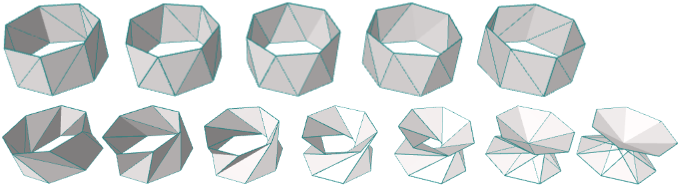

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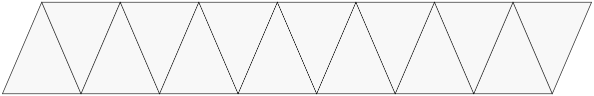

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- Take two ploids with same supporting polygons,
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Diplotorus

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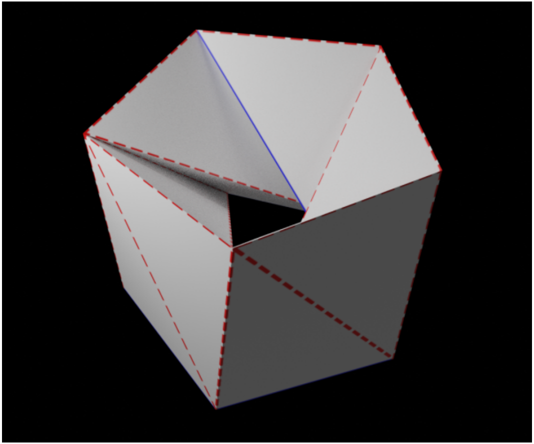
- One can easily compute layouts for a diplotorus



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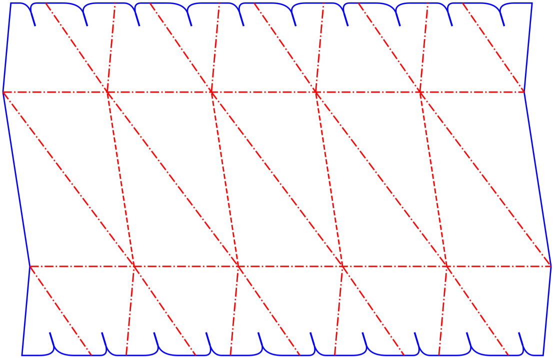

Every torus is a diplotorus

- One can easily find the modulus of a diplotorus
- A bit of computation shows that every flat torus can be realized, giving the result:

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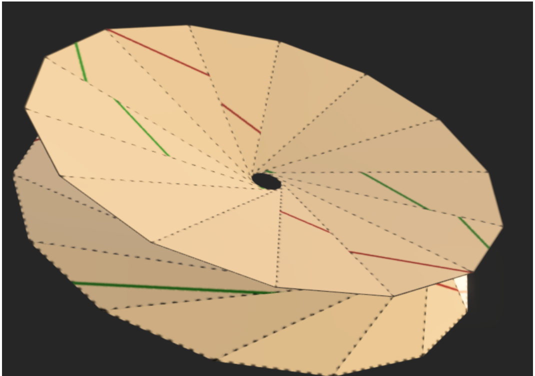

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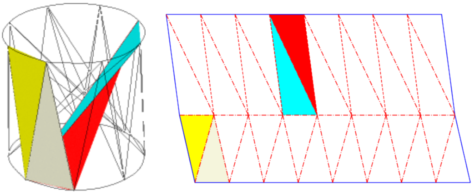
Proof sketch, step 1: Compute a modulus representative out of the diplotori layouts

n: vertices of each regular polygon
h: height of the torus
a: twist parameter of the “inside” ploid
a^*: twist parameter of the “outside” ploid
$d = (a - a^*)/2,\ b = (a + a^*)/2$



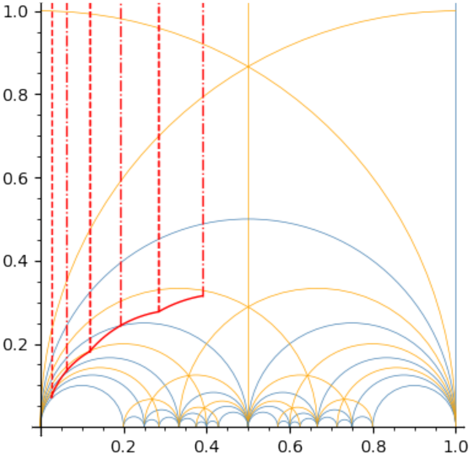
$$\begin{aligned}
m(n,d,a,h) &= m_1(n,d,a)\cdot 1 + m_i(n,d,a,h)\cdot i \quad \text{where} \\
m_1(n,d,a) &= d/n - \cos(b\pi/n)\sin(d\pi/n)/(n\sin(\pi/n)) \\
m_i(n,d,a,h) &= (\sqrt{h^2 + (2\sin((a+1)/2\,\pi/n)\sin((a-1)/2\,\pi/n))^2} \\
&\quad + \sqrt{h^2 + (2\sin((a*+1)/2\,\pi/n)\sin((a*-1)/2\pi/n))^2} \\
&\quad)/(2\,n\sin(\pi/n))
\end{aligned}$$

Proof sketch, step 2: Look at moduli of convex diplotori

A diplotorus is *convex* iff its “outside” ploid is convex.

A ploid is *convex* iff it’s included in the boundary of its convex hull.

(This happens when the twist parameter of the ploid is between 0 and 1.)



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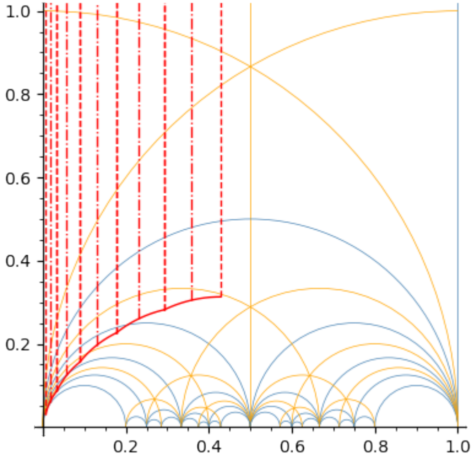

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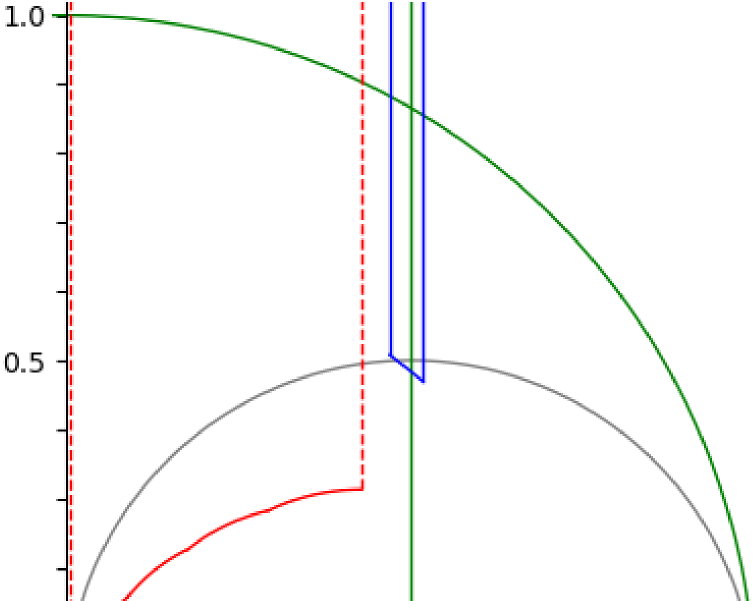
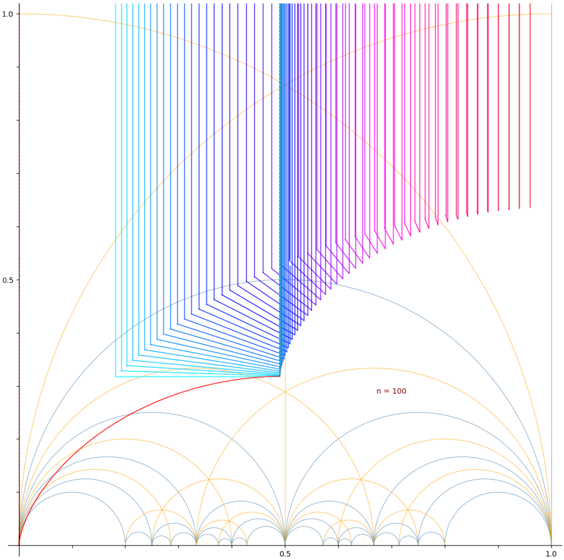
Proof sketch, step 3: Catch the square flat torus (and the torus of the regular hexagon) using non-convex diplotori



Diplotori patches in the hyperbolic plane



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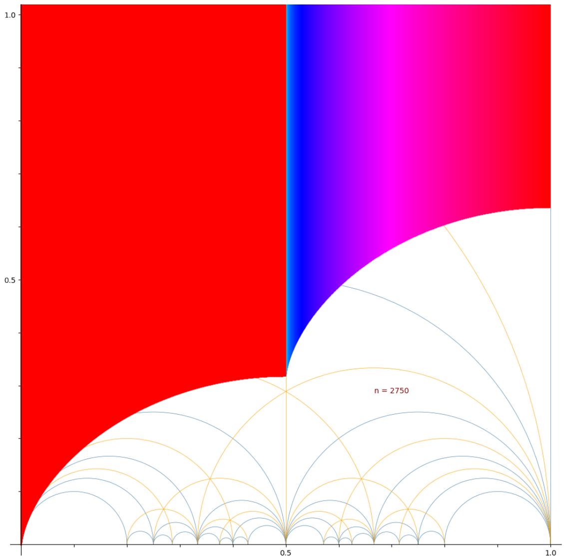

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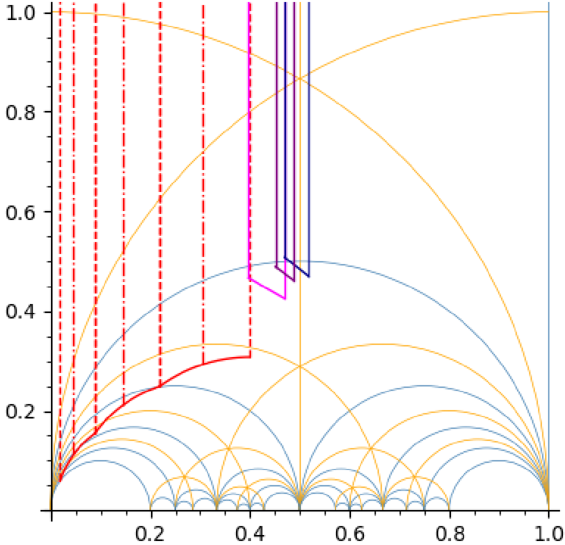

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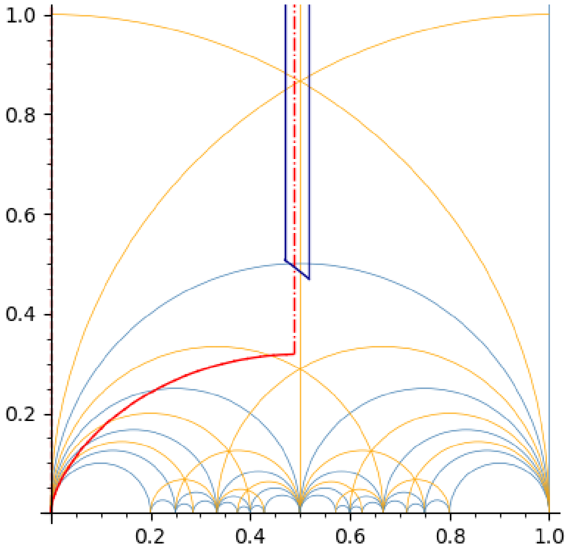

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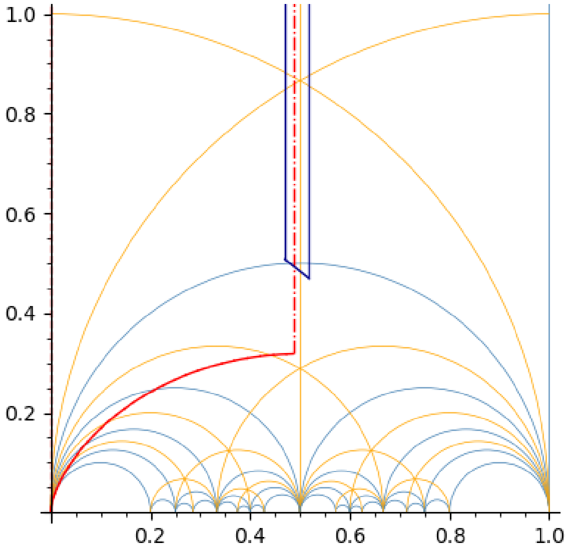

Florent Tallerie: fill in the “hole” with Zalgaller’s “long tori”

Symmetrization of diplotorus

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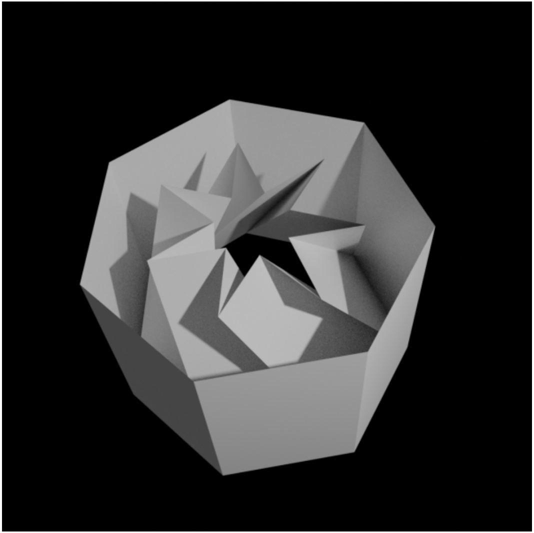

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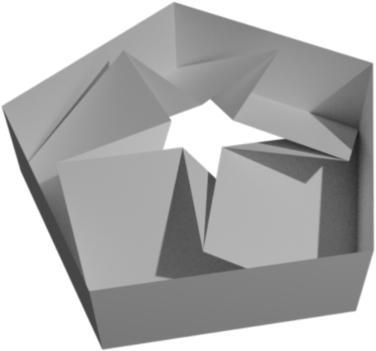

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Translation surfaces

- Glue diplotori along square faces:
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- Question: describe the translation surfaces obtained.
- All our examples have cone angles 4π.

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