Polyhedral flat tori

Alba MÁLAGA (j.w. S. Lelièvre and P. Arnoux)

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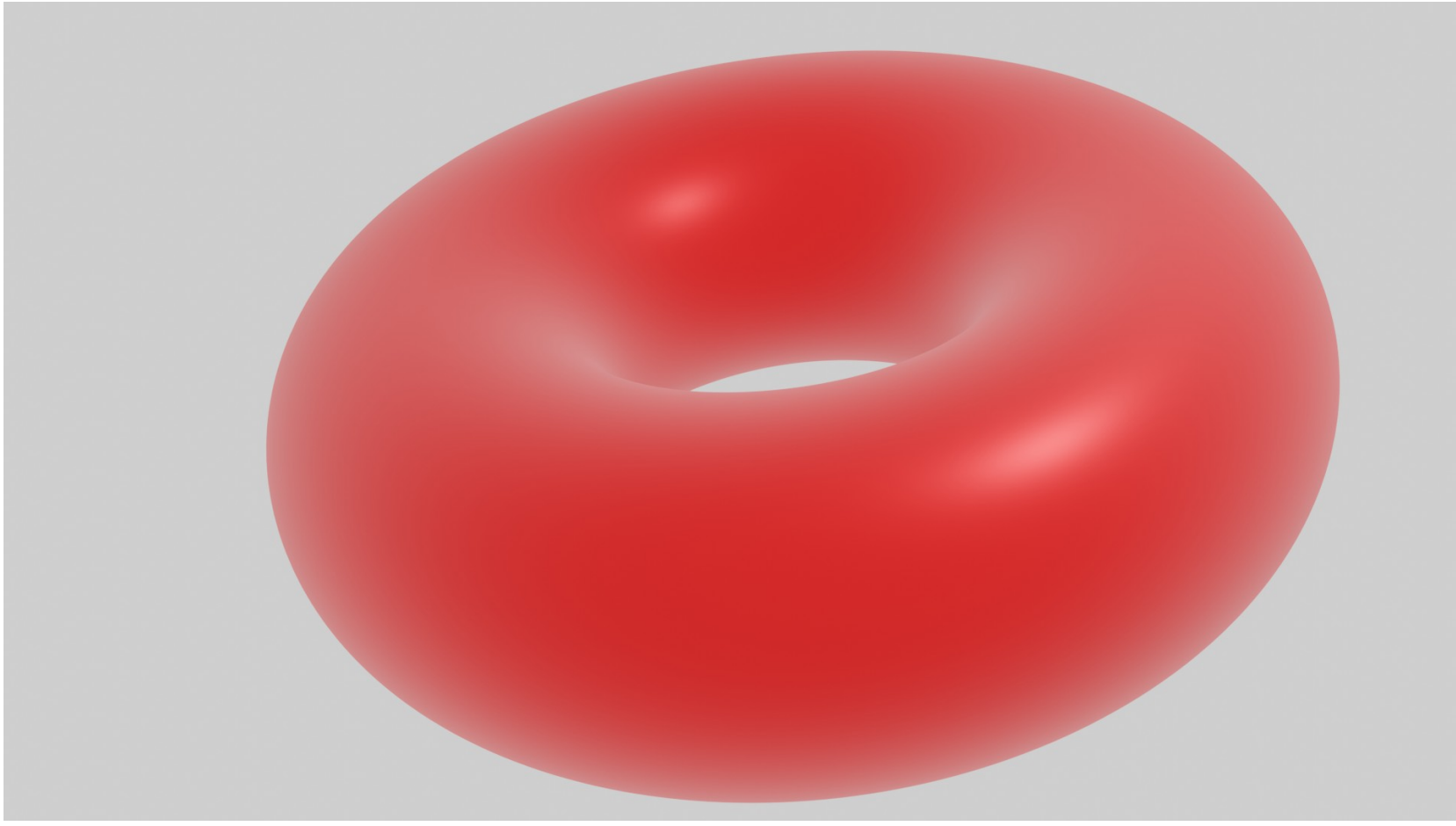
What is a torus

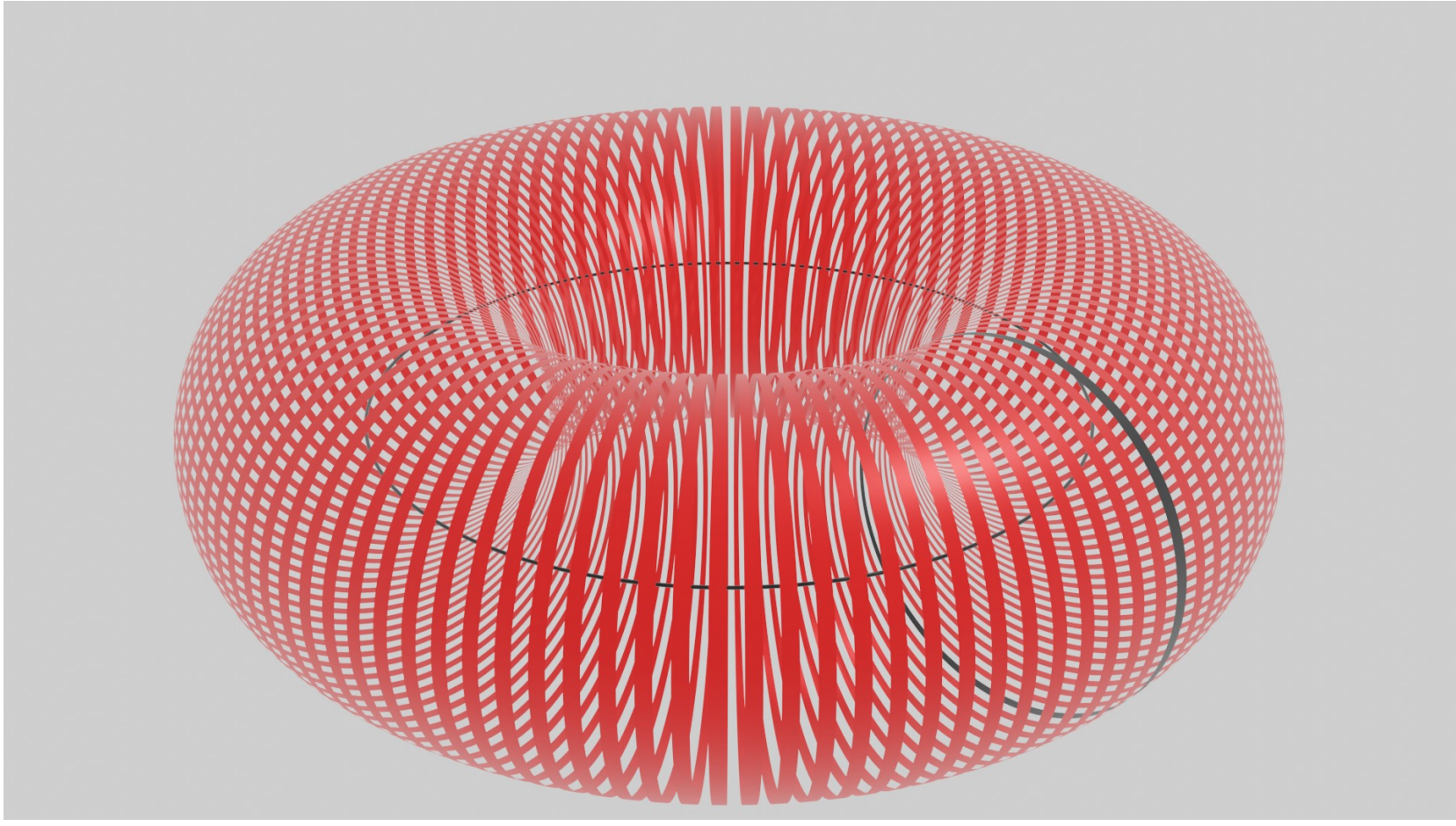
A few examples

What do these objects have in common?



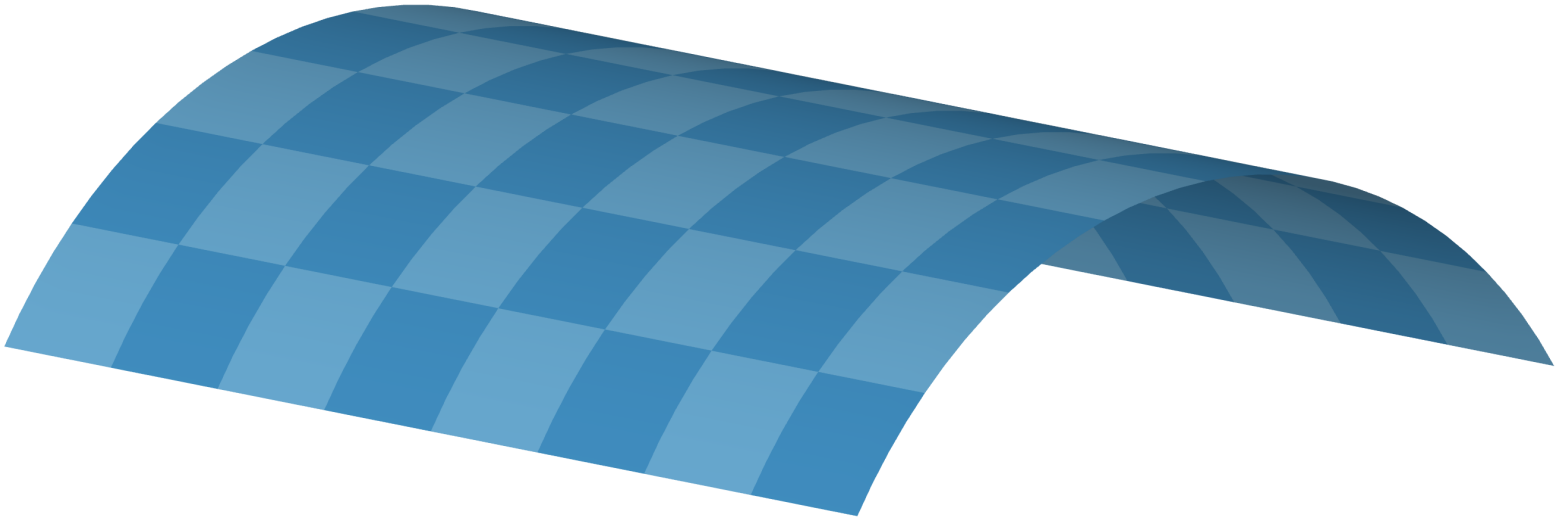
Any space homeomorphic to the round torus is called a torus.

For example, the cartesian product of two circles is a torus, because we obtain the round torus by sliding a smaller circle along a larger circle.



What is a flat torus?

Product of two circles: also a square with opposite sides glued.



Gluing opposite sides of a square by translation gives the **square flat torus**.

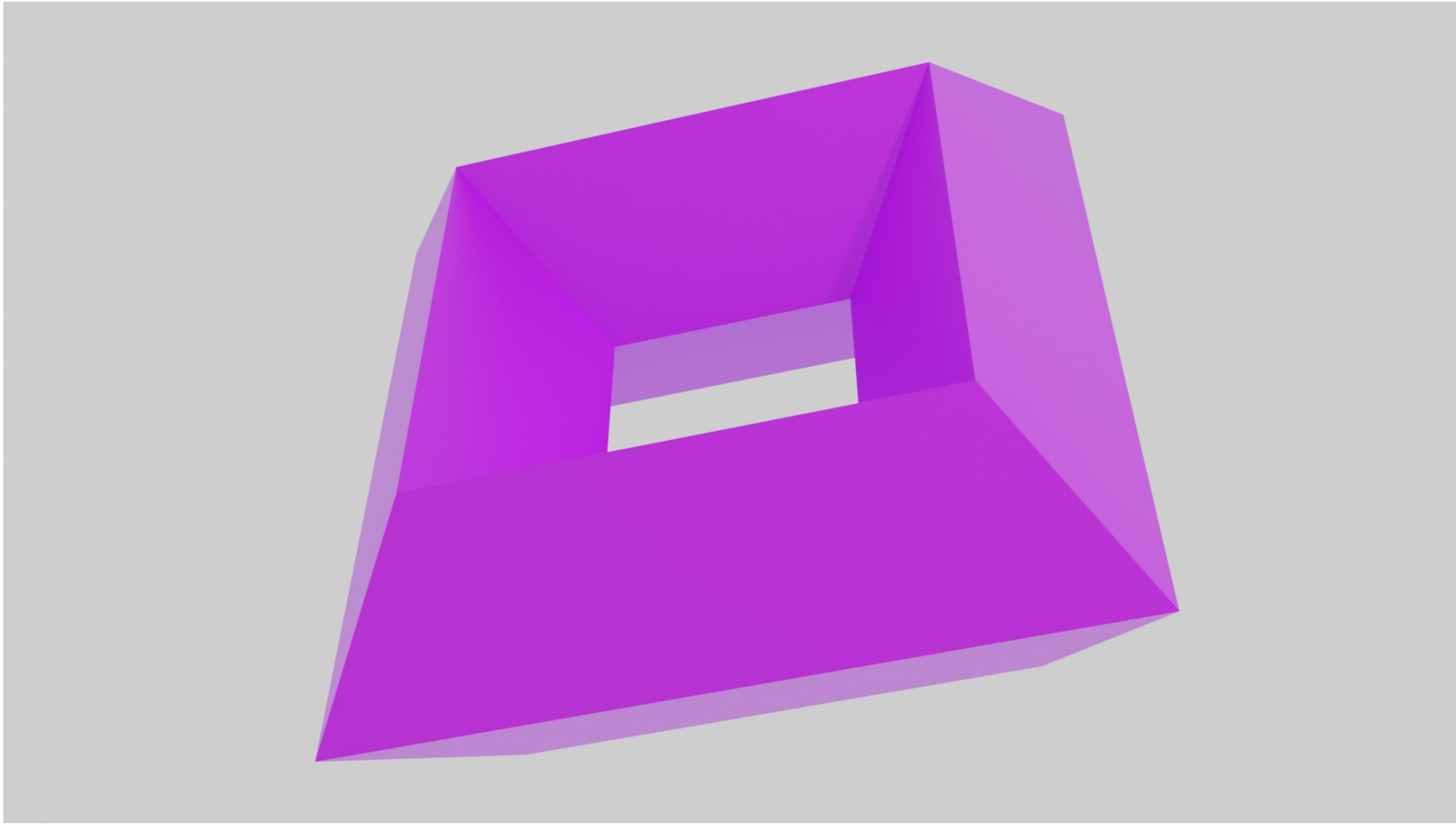
Its intrinsic geometry has constant zero curvature: it is everywhere flat.

More generally, a **flat torus** is a torus endowed with a metric that makes it everywhere flat.

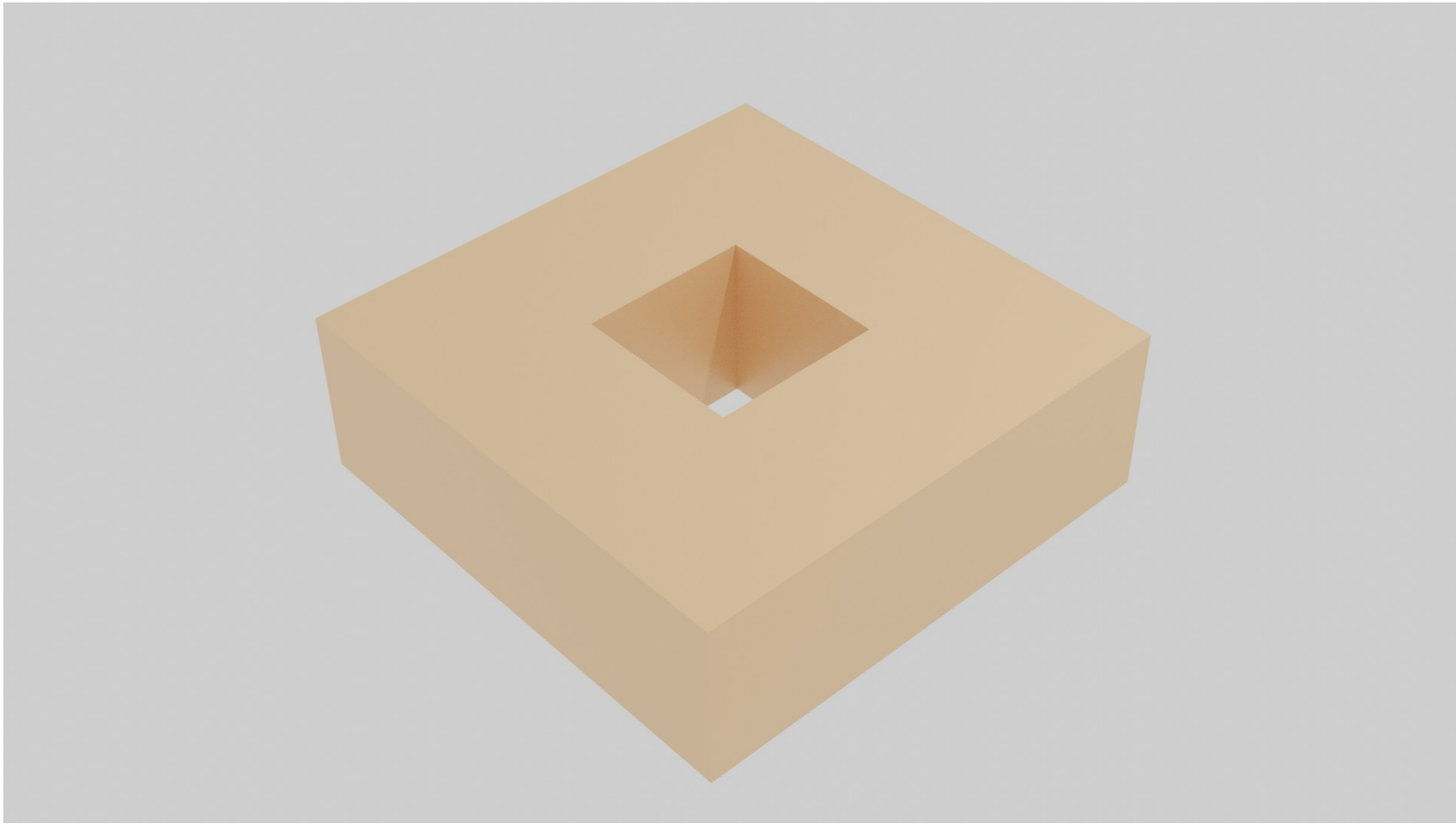
All such tori are parallelograms with opposite sides glued by translation.

Is a playdough torus a flat torus?

Is this “diamond” torus a flat torus?



Is this “square torus” a flat torus?



Question: Can flat tori be realized in Euclidean space?

They **CANNOT** occur as smooth closed compact surfaces! : (

They **CAN** be realized if we relax the smoothness requirement. :)

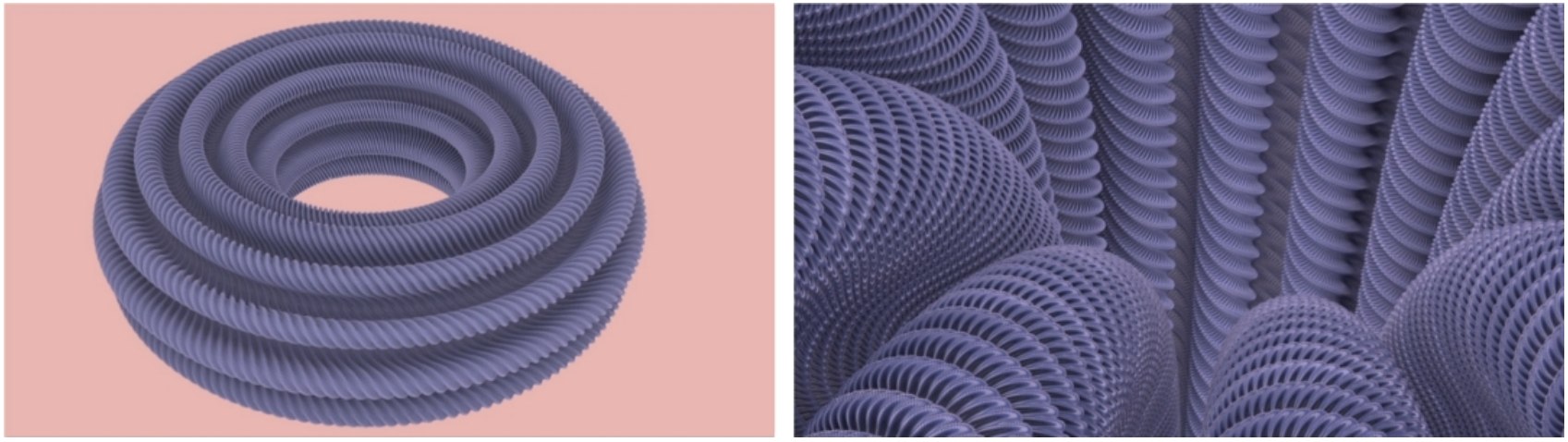
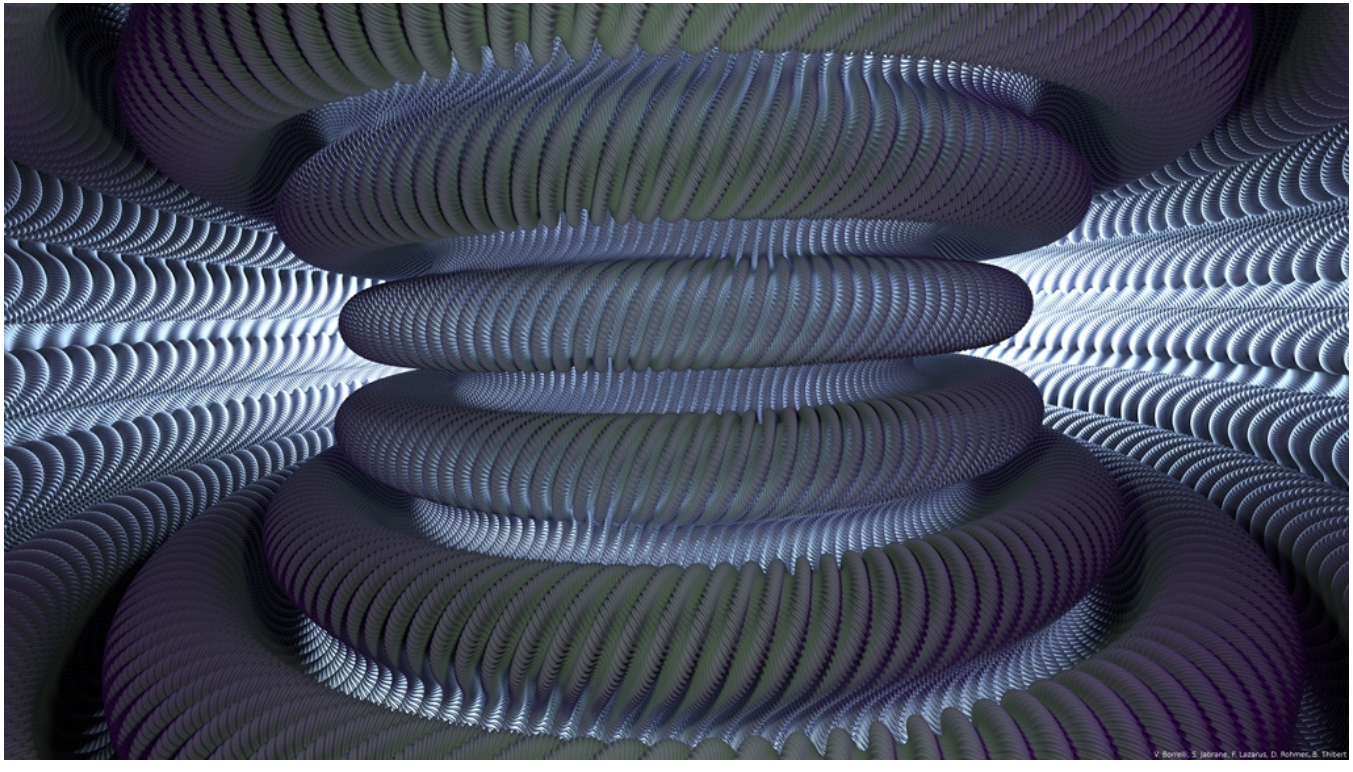


Figure 2: Hevea's corrugated C^1 torus, images by Borrelli, Jabrane, Lazarus, Thibert and the Hévéa team



Flat polyhedral tori

Burago & Zalgaller's construction

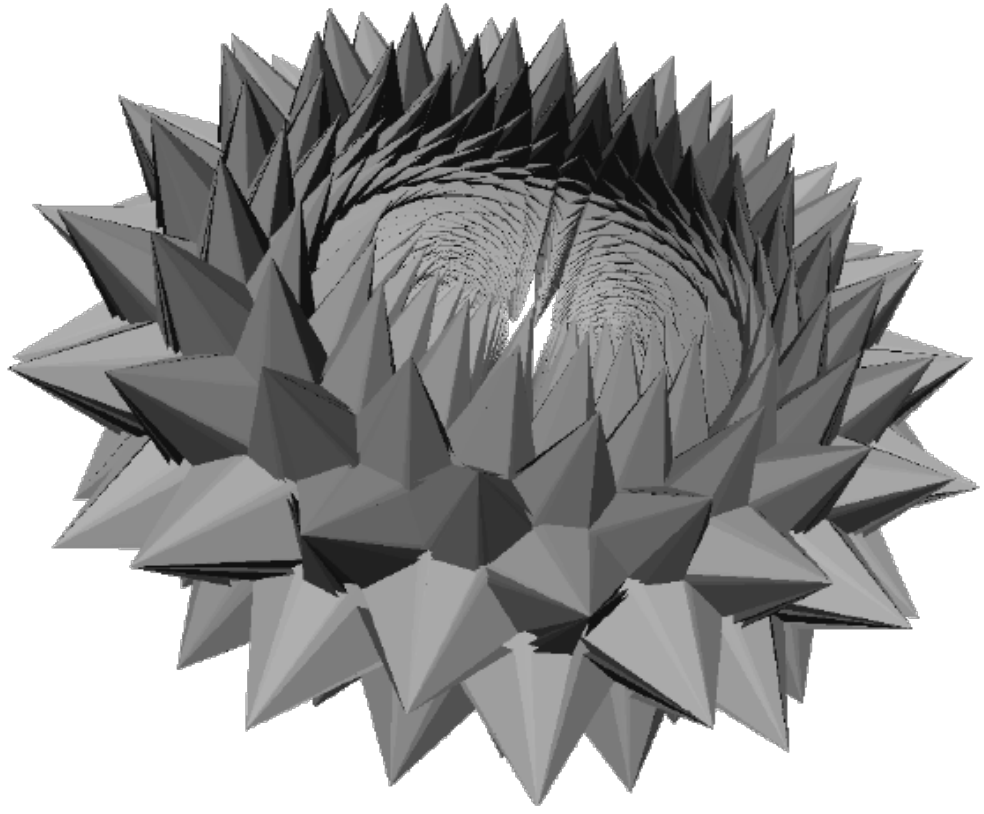


Figure 4: Polyhedral flat torus following the method by Burago and Zalgaller, by Tallerie

Zalgaller's long tori

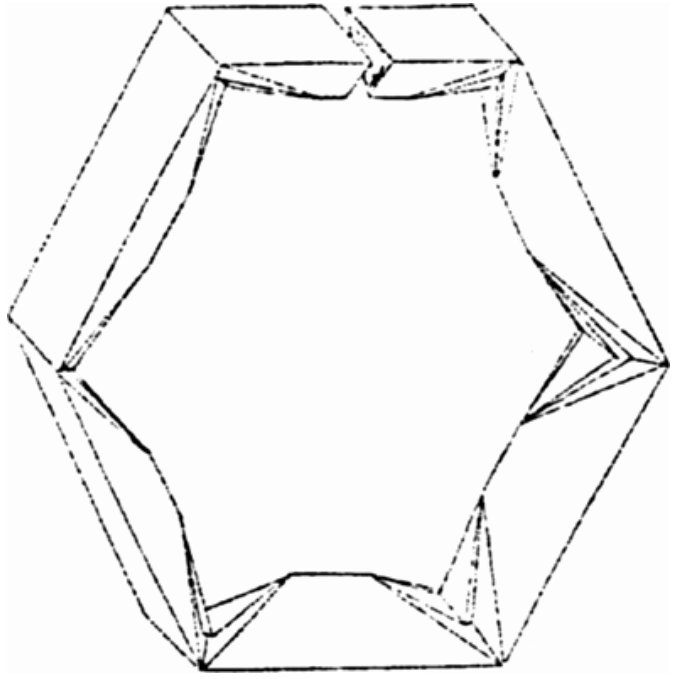


Figure 5: Torus glued from a very long parallelogram (here a rectangle) by Zalgaller

Quintanar's finite corrugations

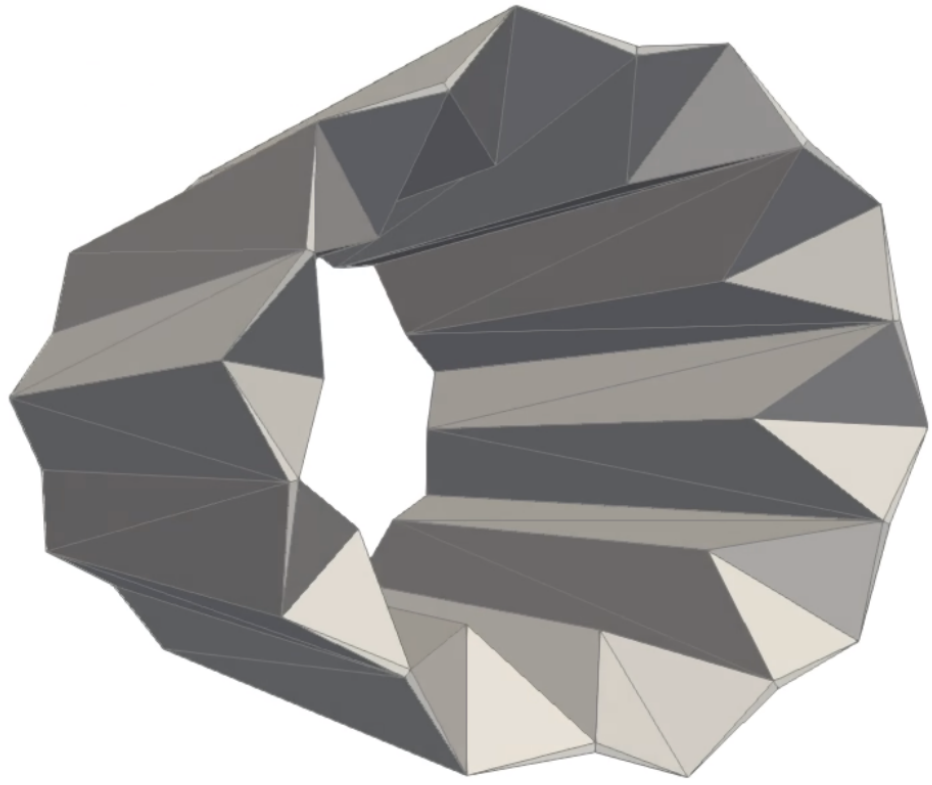
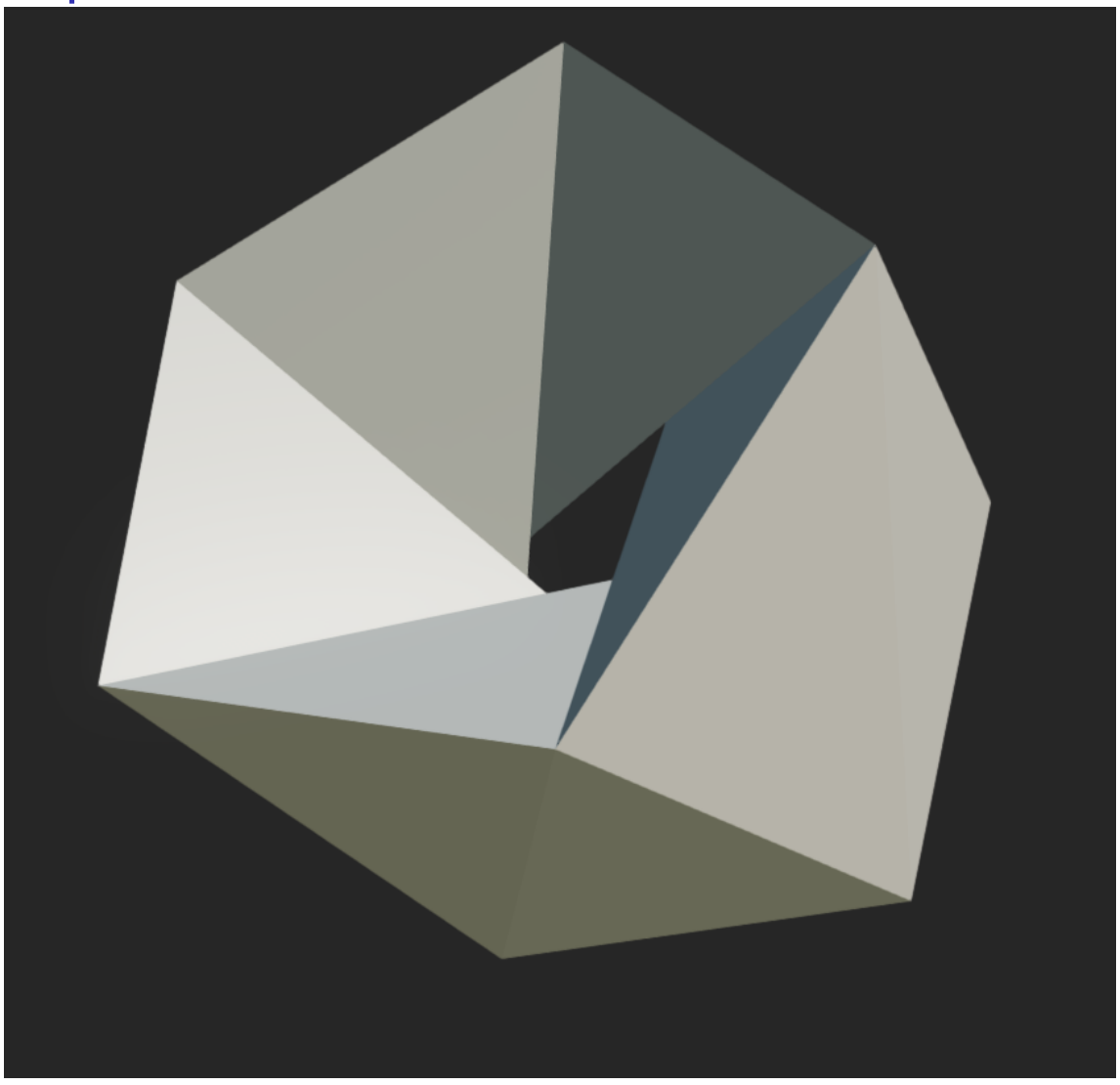


Figure 6: A polyhedral square flat torus by Quintanar

Diplotori



A family of polyhedral flat tori with $2n$ vertices ($n \geq 5$). Also studied by Tsuboi.

Diplotori as diploid tori

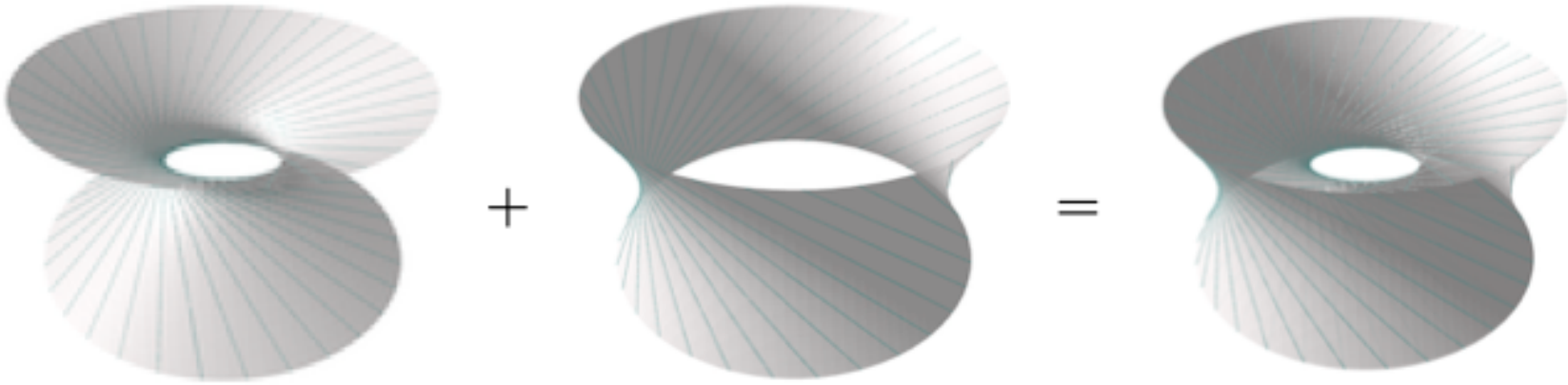


Figure 7: The union of two hyperboloids with same supporting circles is a torus.

Diplotori as diploid tori

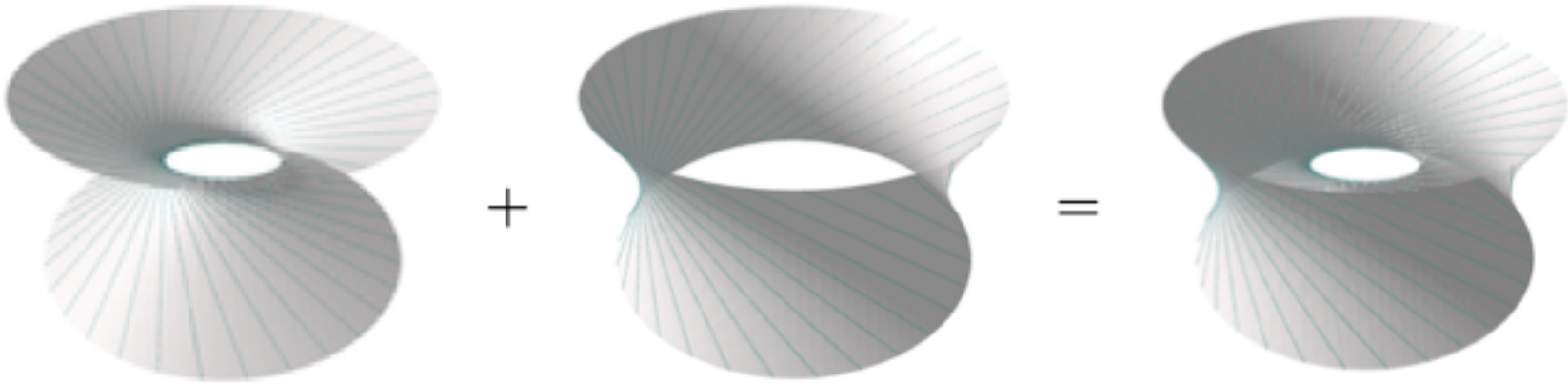


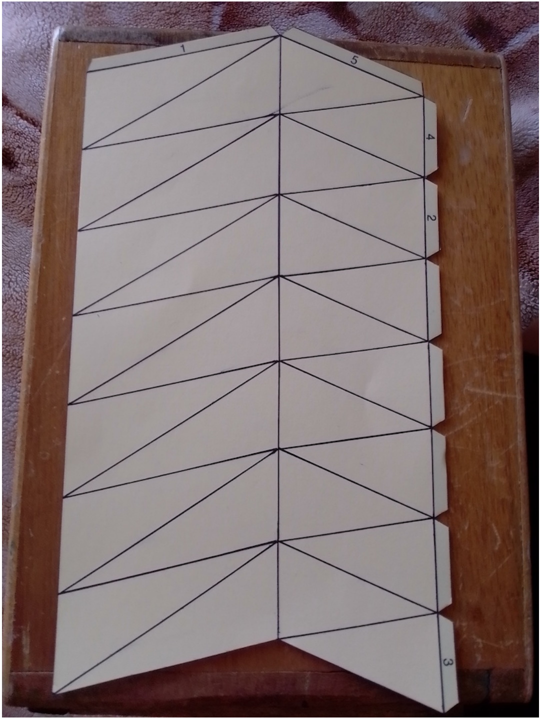
Figure 7: The union of two hyperboloids with same supporting circles is a torus.

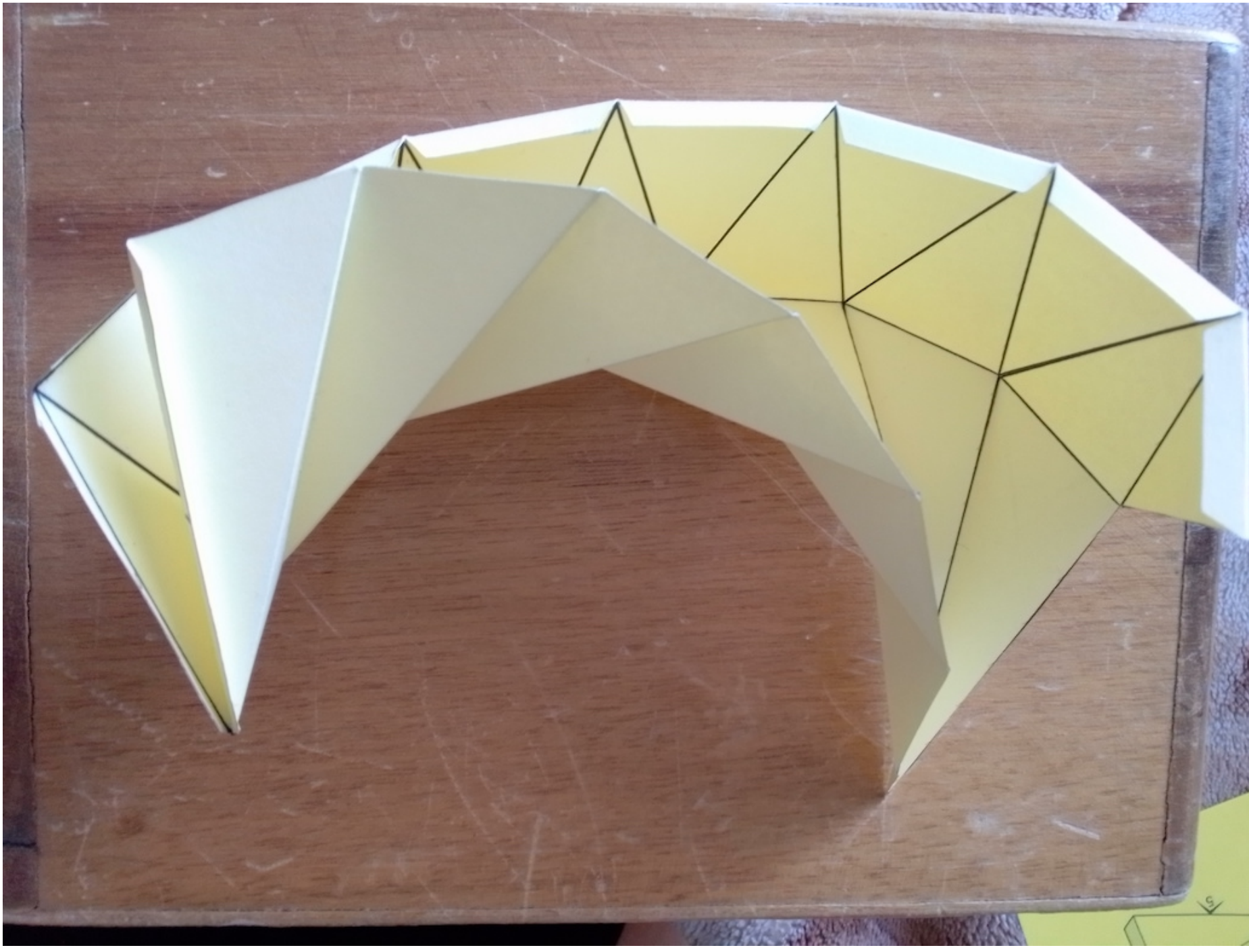
Figure 8: The union of two ploids with common bounding polygons is a

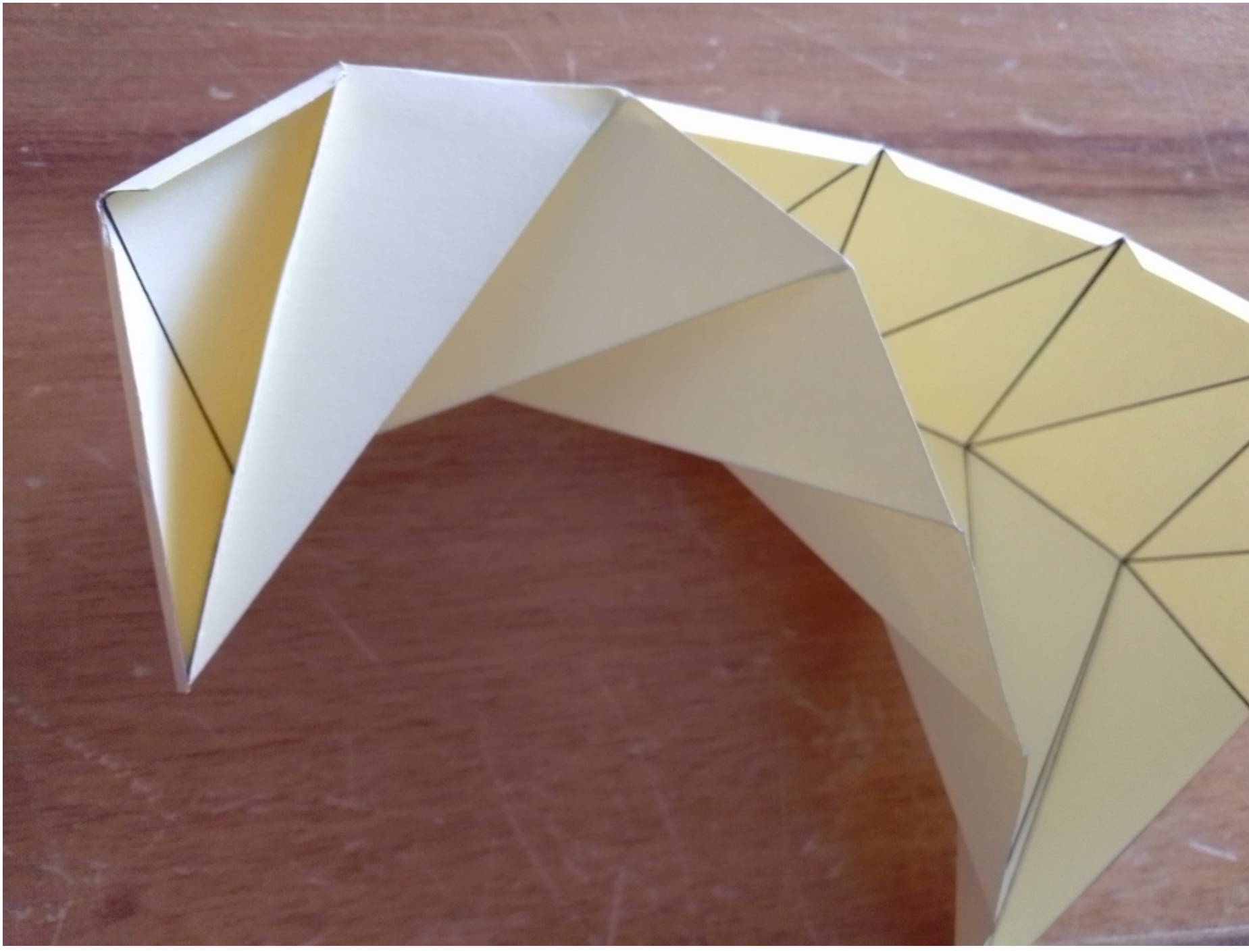
OK but how to BUILD such a torus in practice?

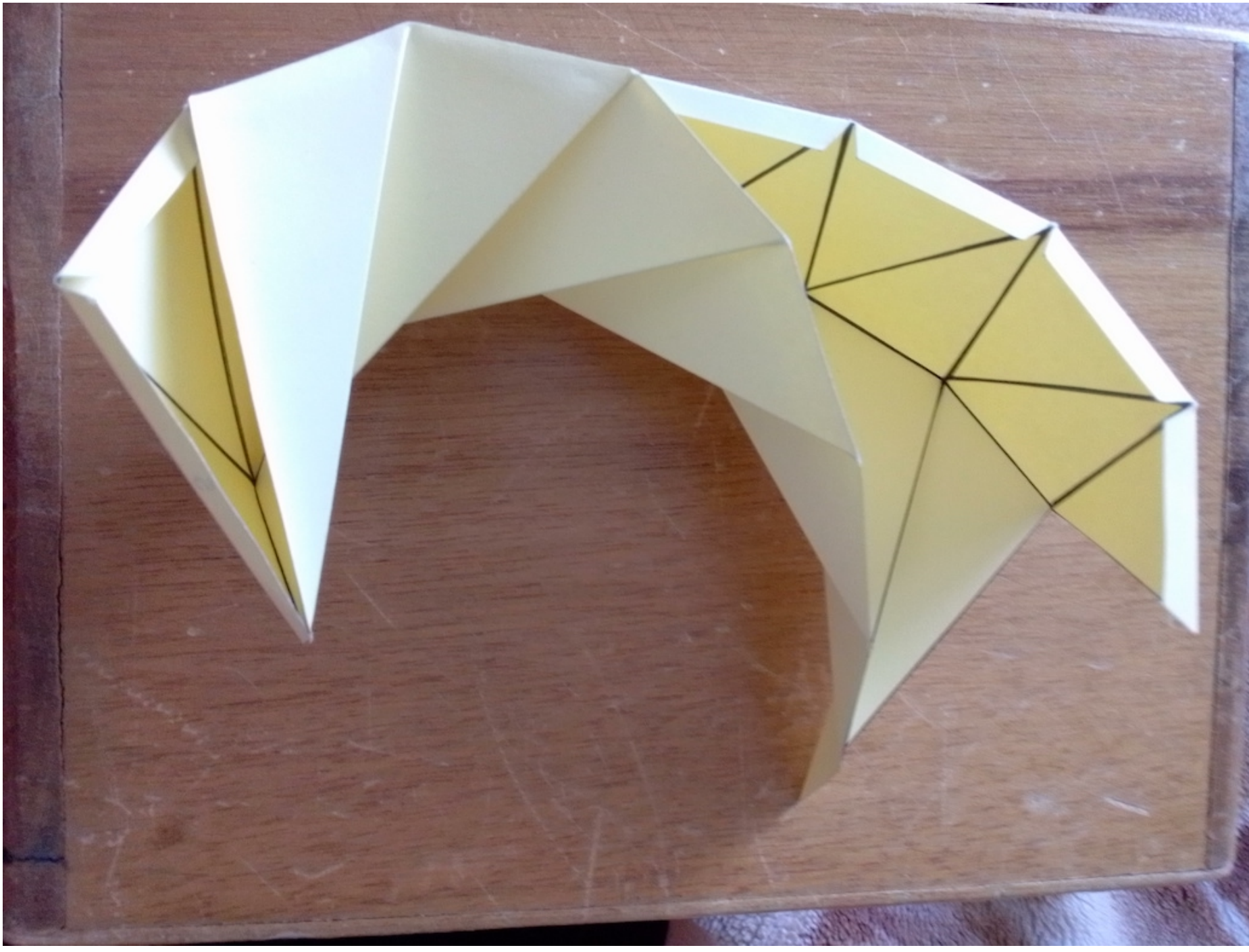
It's a polyhedron! It is flat!

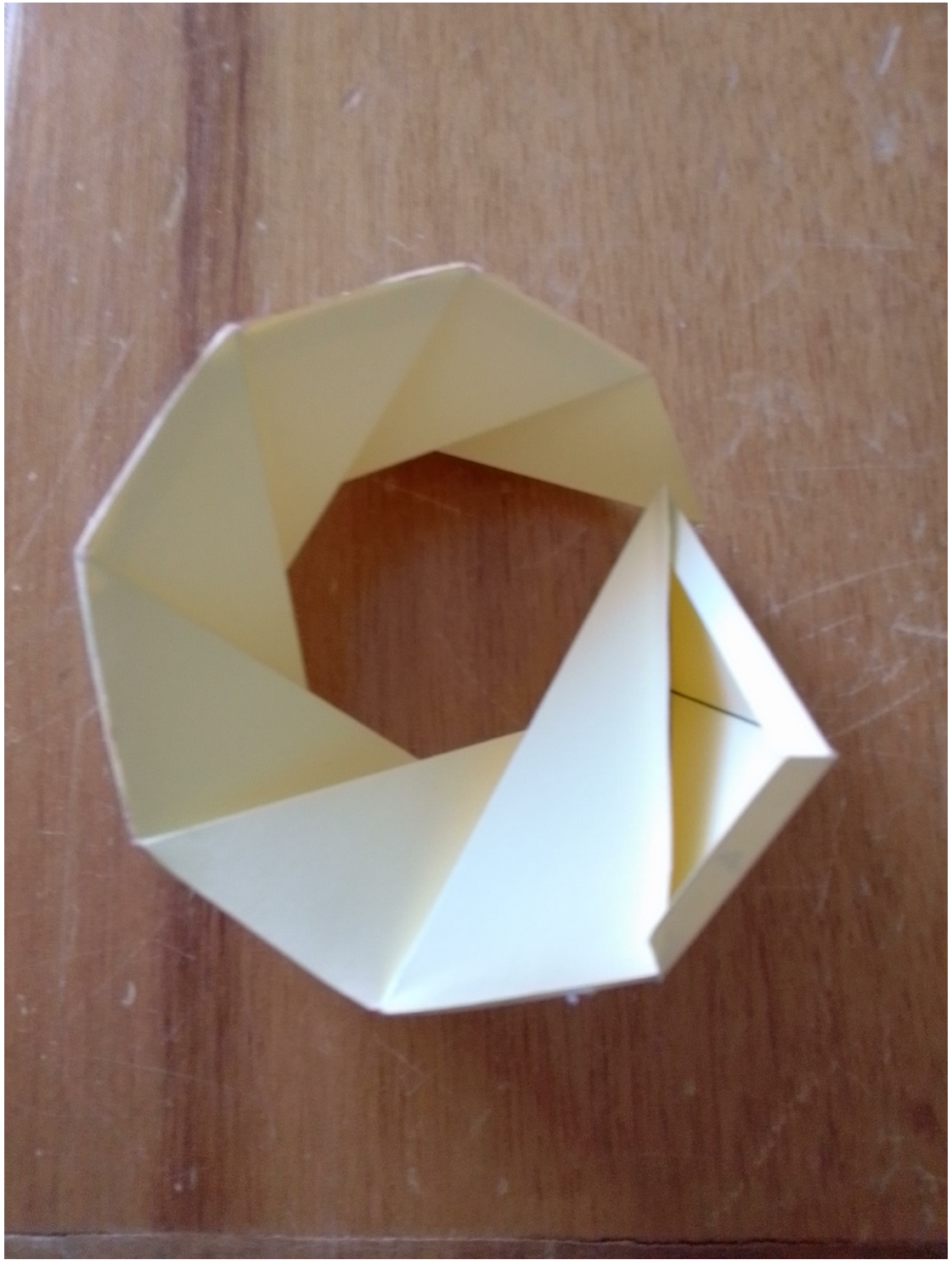
So: cut paper, fold, and glue.

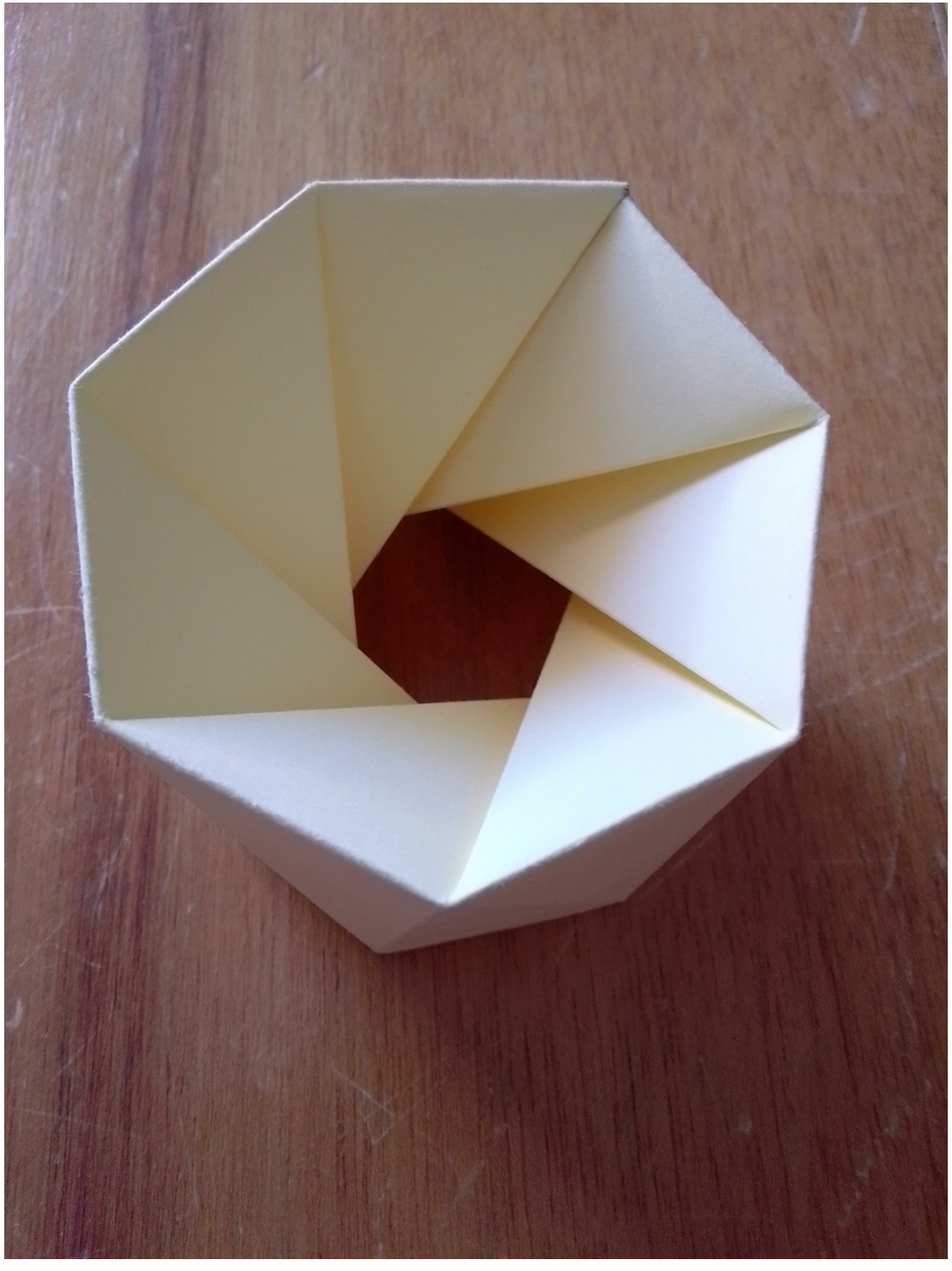

1
5
4
2
3











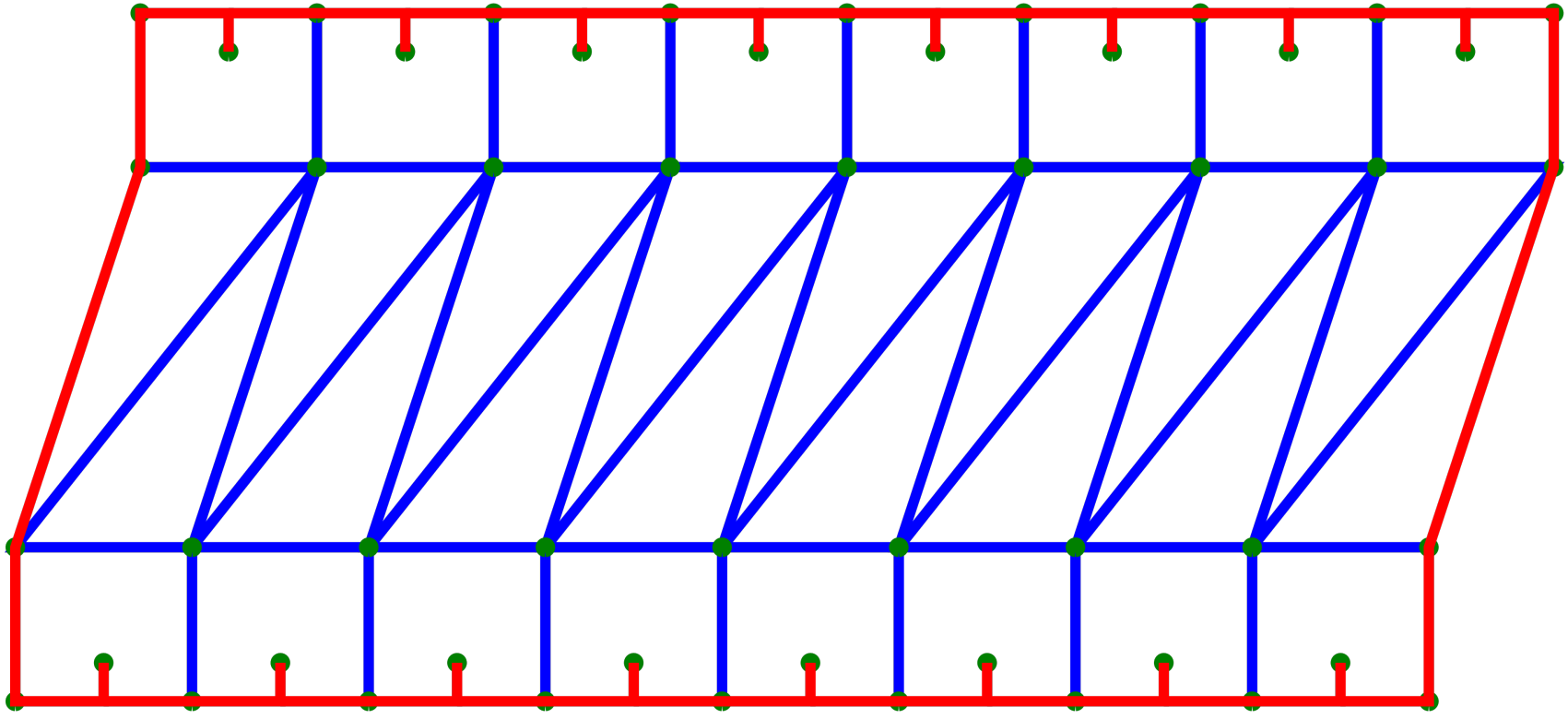


Figure 15: Flat torus layout

Or use one of our glueless layouts, soon to appear at kits.math.cnrs.fr and imaginary.org

Question: How diverse are the flat tori that can be obtained that way?

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? ? ? ? ? ? ? ? ? ? ? ? ? ? ?

How to think about the space of flat tori?

We do not want to distinguish between tori scaled from each other.

The layout for a flat torus, if rotated or translated, will give the same torus.

The modular curve of flat tori

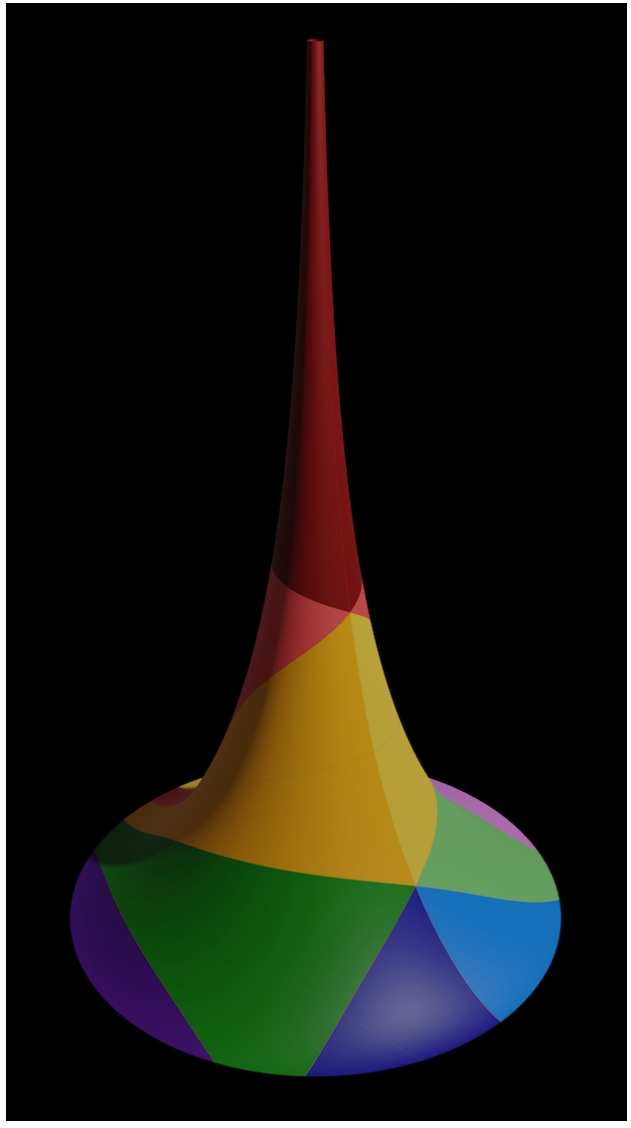
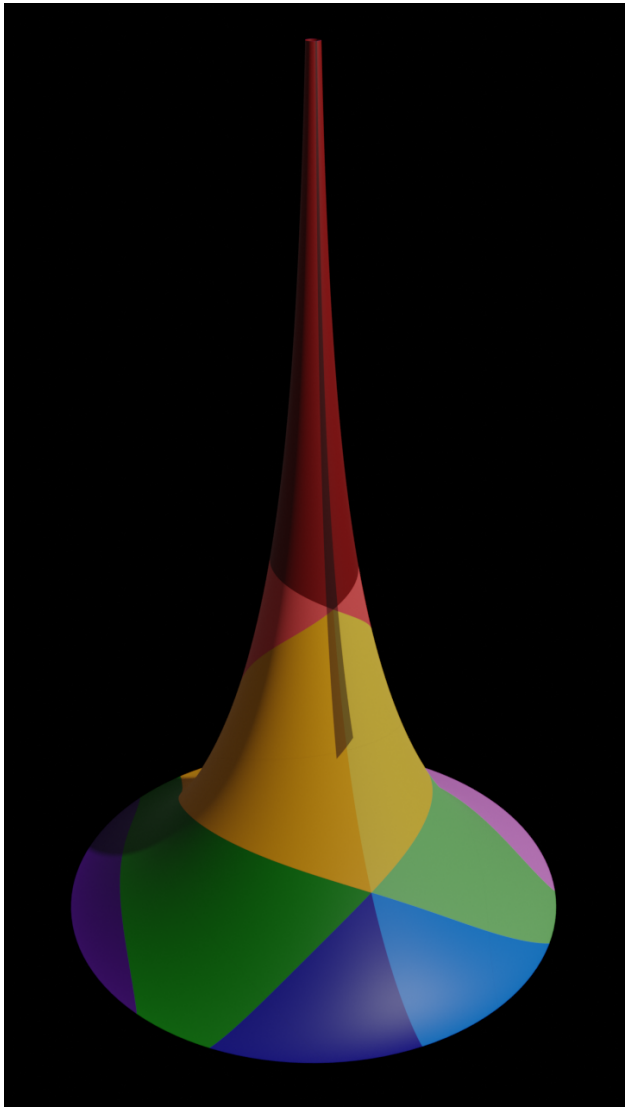


Figure 16: Modular curve of tori, at the pseudosphere

Question: Are all points of the modular curve diplotori ?

Theorem (MS, Arnoux, Lelièvre; Tsuboi): **YES !**



Some history

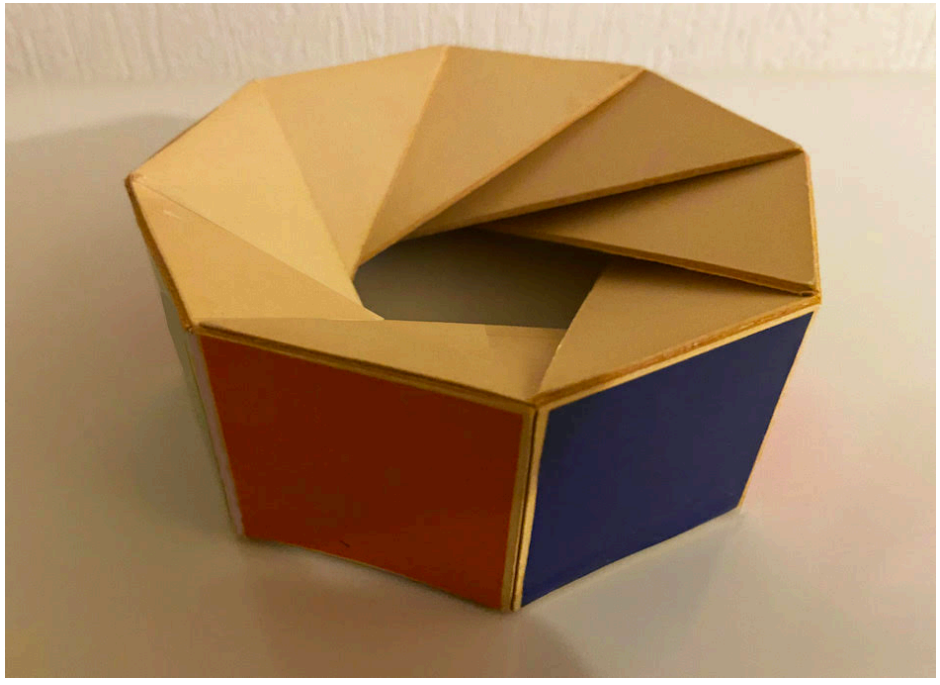


Figure 17: Diplotorus made by Guy Valette in 1984

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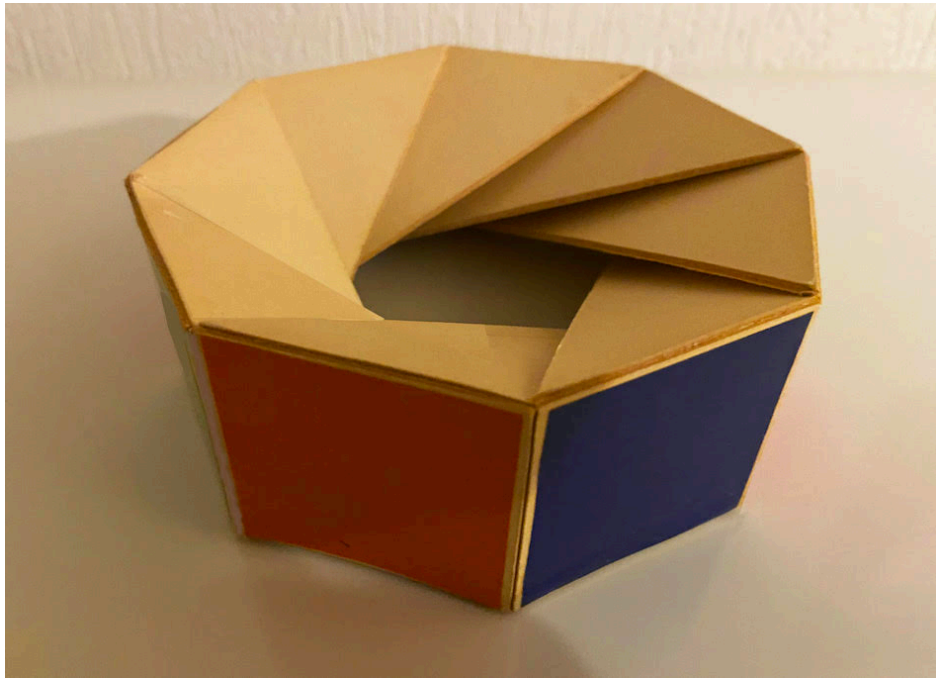

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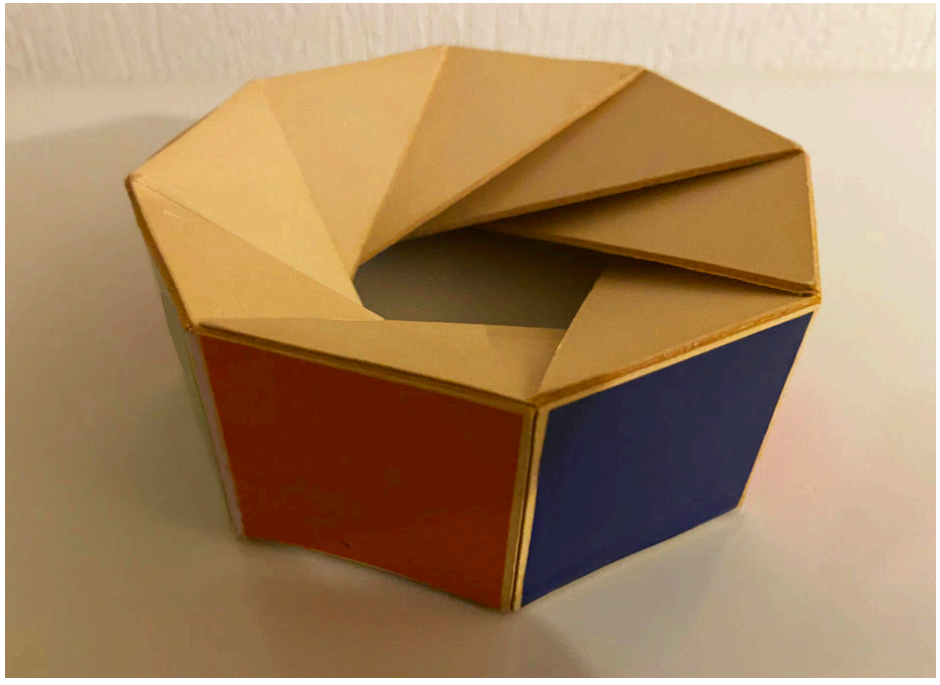

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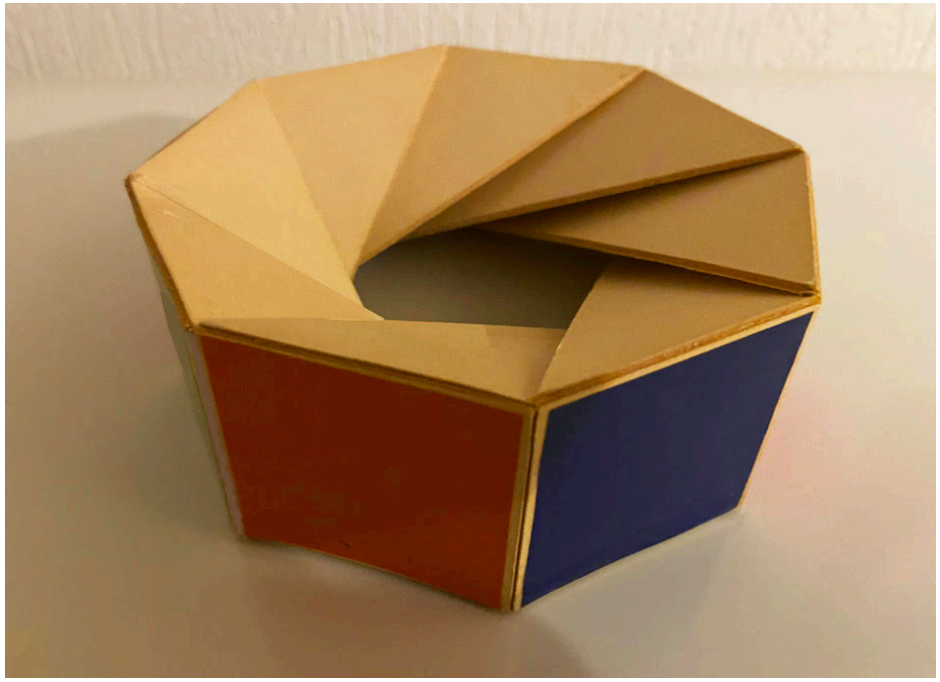

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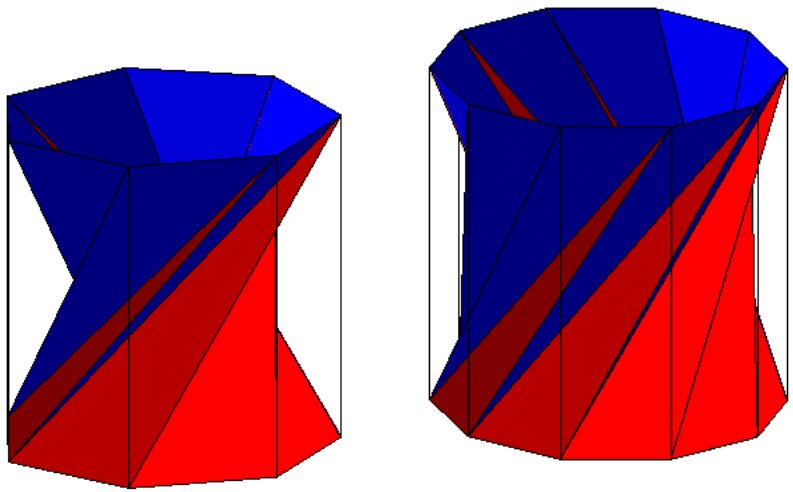
- Segerman saw Ferréol's MathCurve page in 2015, told us in 2019
- Valette had told Ferréol after hearing about it at a meeting in Oberwolfach in July 1984
- interest in these objects likely goes back to the 1960s

Works on this topic

- Yu. D. Burago; V. A. Zalgaller. Isometric piecewise linear immersion of two-dimensional manifolds with polyhedral metrics into $\mathbb{R}^3$. Algebra Analiz, vol 7, no 3, 76-–95 (1995). English translation: St Petersburg Math J., vol 7, no 3, 369–385, 1996. https://zbmath.org/0851.52018
- Takashi Tsuboi. On origami embeddings of flat tori. arXiv:2007.03434

- Robert Ferreol. Tore plat. “MathCurve.com” website. http://mathcurve.com/polyedres/toreplat/toreplat.shtml



- Joseph O’Rourke. On Folding a Polygon to a Polyhedron. 2010. https://arxiv.org/abs/1007.3181

- Patricia Tanessi Quintanar Cortés. Polyhedral embeddings of the flat square torus. PhD thesis, Université Lyon 1. 2019. https://tel.archives-ouvertes.fr/tel-02613763
- Henry Segerman. Visualising mathematics with 3D printing. pp 127 sqq. http://www.3dprintmath.com/figures/6-12/

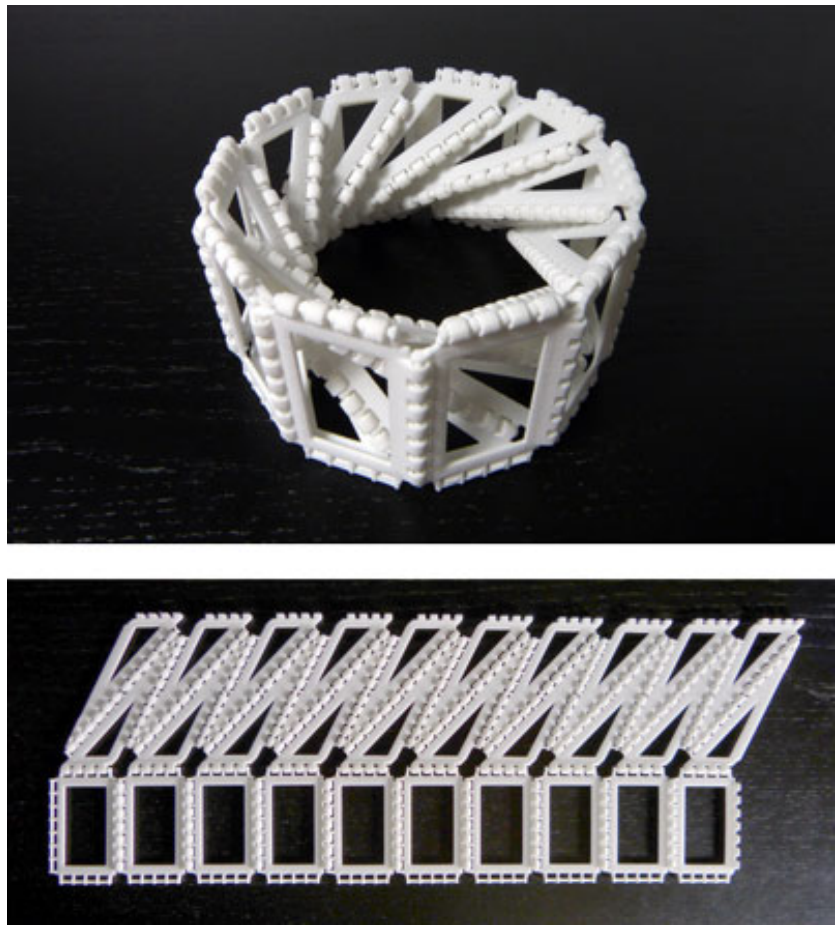


Figure 18: Hinged flat torus by Henry Segerman

- V. A. Zalgaller. Some bendings of a long cylinder. Journal of Mathematical Sciences, 100:3, 2228–2238, 2000. (Translated from a 1997 article published in Russian in Zapiski Nauchnykh Seminarov POMI) http://mi.mathnet.ru/eng/znsl549 http://mi.mathnet.ru/eng/znsl/v246/p66 https://link.springer.com/article/10.1007%2Fs10958-000-0007-3
- Question 208996 on Math Overflow. https://mathoverflow.net/q/208996
- Tutorial for a polyhedral flat torus from a decagon. "Instructables". https://www.instructables.com/id/Tutorial-10-Collapsible-Paper-Tower-Helix-Decagon/
- Alba Málaga, Flat tori: Geogebra animation. https://www.geogebra.org/m/ttd9f4bb