Computing complete hyperbolic structures on cusped 3-manifolds

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Knots

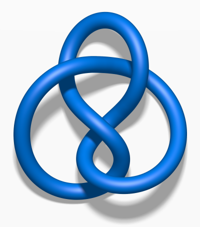

Figure-eight knot[1]

Ambient isotopy

Continuous distortion of the ambient space.

[1] https://en.wikipedia.org/wiki/Figure-eight_knot_(mathematics)

A brief history of knot censuses

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With several mistakes on the way. . .

3-Manifolds and knots

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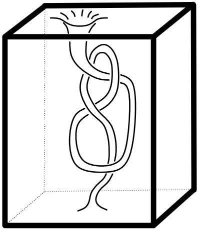

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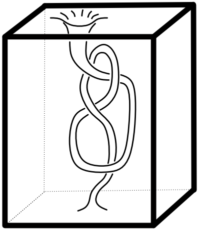

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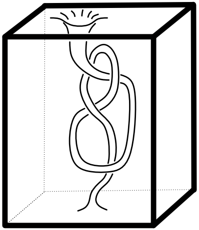

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Theorem (Thurston)

Knots are either satellites, torus or hyperbolic.

Complete hyperbolic structures

Provides a hyperbolic metric on the manifold.

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Mostow rigidity

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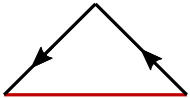
The hyperbolic volume

Great invariant linked to many theories and conjectures.

Outline

Generalized triangulations

Triangulated 3-manifolds with self-identifications.



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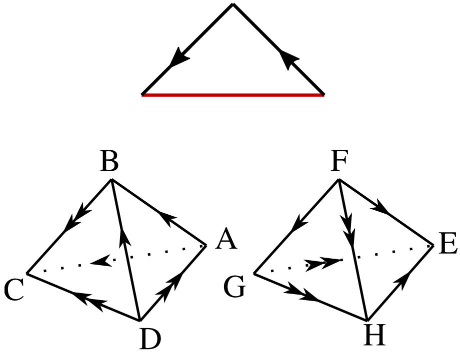
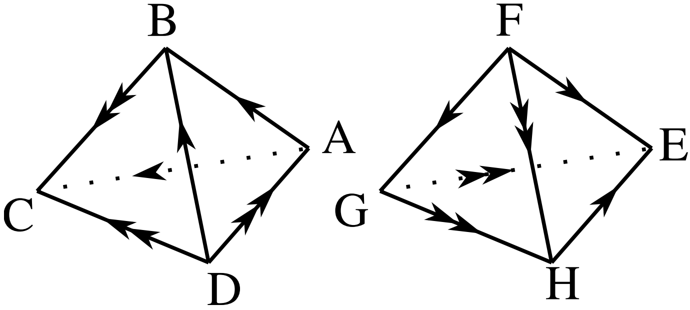


Knot complement triangulations



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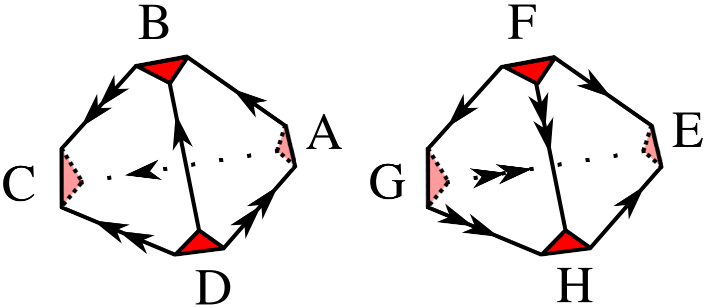

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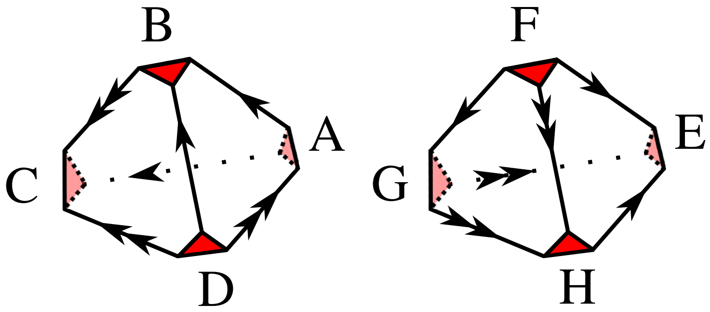
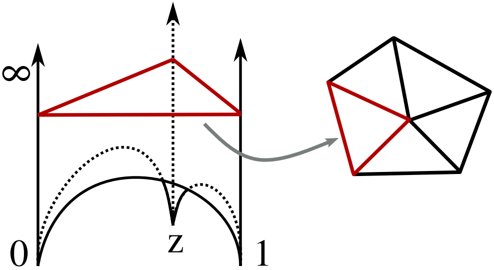


All 3-manifolds are triangulable, and there exists an algorithm for knots complements.

Hyperbolic ideal tetrahedra



Definition

A hyperbolic ideal tetrahedron is the convex hull of four distinct points on $\partial\mathbb{H}^3$.

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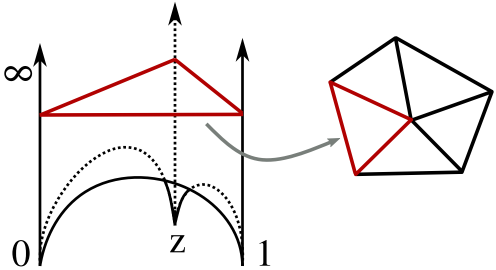

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Either described with a single complex parameter or three dihedral angles.

Getting a hyperbolic manifold

Every point must have a neighborhood isometric to a sphere.

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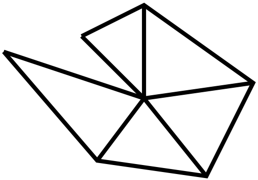

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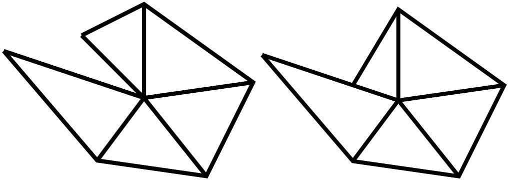

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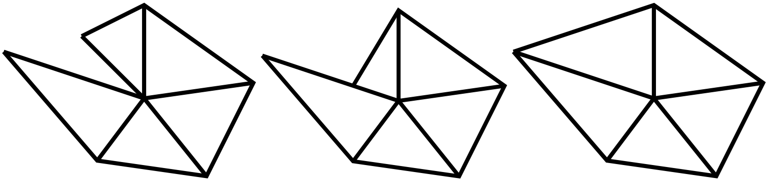
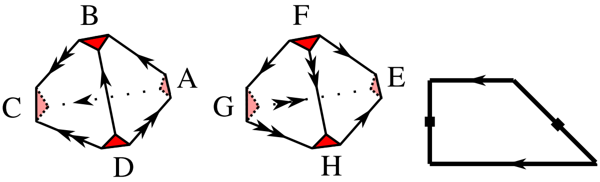


Completeness around the cusp



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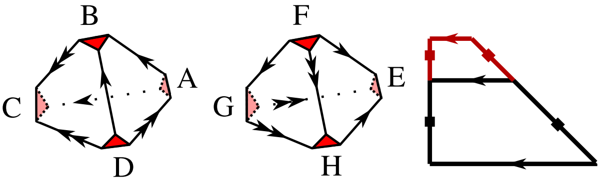

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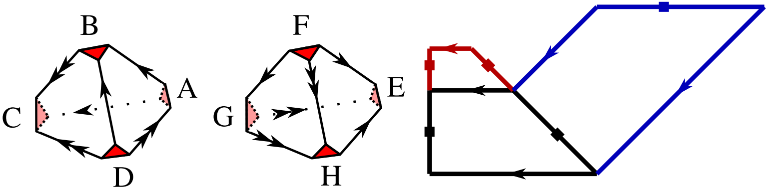

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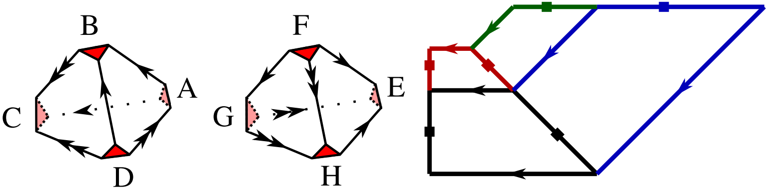

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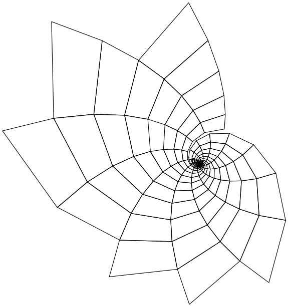

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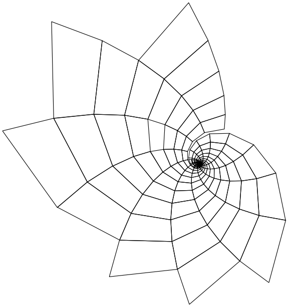

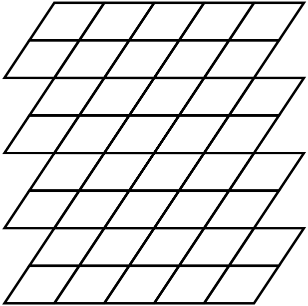

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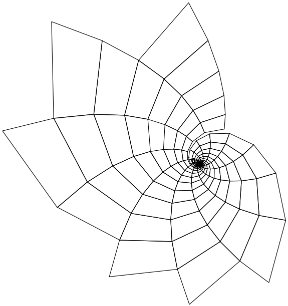
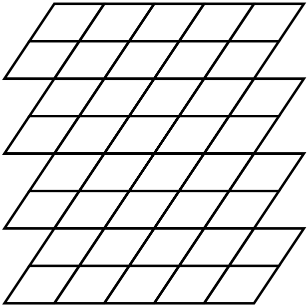




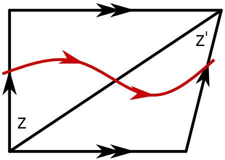
Theorem

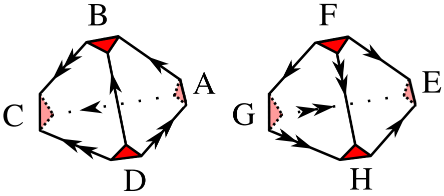
The hyperbolic metric of the 3-manifold is complete if and only if the euclidean metric on the boundary torus is complete.

Normal curves and holonomy

Normal curve

A sequence of segments cutting the triangles only by their edges.





Thurston gluing equations (1980)

Edge equations

$$\sum_i \log(z_i) = 2i\pi$$

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Complete hyperbolic structure problem

- Input: triangulation τ of a knot complement.
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Complete hyperbolic structure problem

- Input: triangulation τ of a knot complement.
- ~~Output: complete hyperbolic structure on τ.~~
- Output: a triangulation equivalent to τ admitting a complete hyperbolic structure.

SnapPy

Uses Newton's method to directly solve Thurston's equations.

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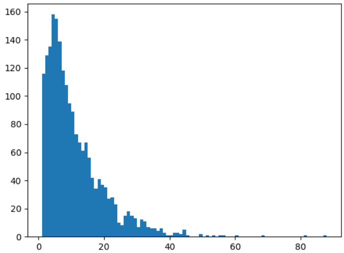
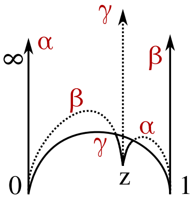
Uses Newton's method to directly solve Thurston's equations.

- No guarantees on the convergence speed.
- No studies of the failures cases.
- No methods/heuristics to find geometrizable triangulations.



The polytope of angle structures



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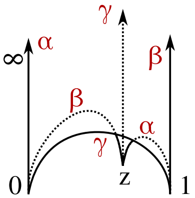

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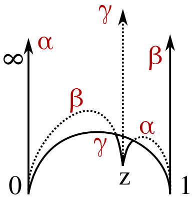

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Lemma (Neumann, 1992)

With τ the triangulation and $\mathcal{A}(\tau)$ the polytope of angle structures:

$$dim\ \mathcal{A}(\tau) = |\tau| + |\partial M|$$

Existence of a hyperbolic metric

Theorem (Casson)

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A point $p \in \mathcal{A}(\tau)$ corresponds to a complete hyperbolic metric on the interior of M if and only if p is a critical point of the volume functional $\mathcal{V} : \mathcal{A}(\tau) \to \mathrm{R}$.

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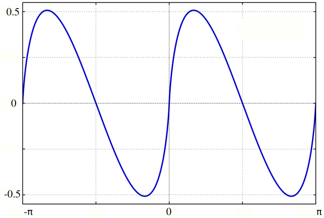
Strategy $\to$ maximize $\mathcal{V}$ over $\mathcal{A}(\tau)$.

Volume of an angle structure

Lobachevsky function

$$Л(x) = -\int_0^x \log|2\sin t|\,\mathrm{d}t$$



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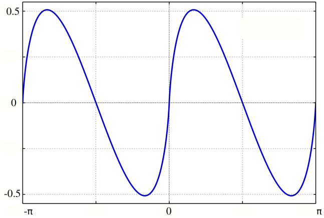

Volume of an ideal tetrahedron

$$\mathcal{V}(\alpha,\beta,\gamma) = Л(\alpha) + Л(\beta) + Л(\gamma)$$

Convexity

Optimization problem

- Maximize $\mathcal{V}$ over $\mathcal{A}(\tau)$.
- A base of $\mathcal{A}(\tau)$ is can be easily computed.
- $\mathcal{V}$ is the sum of the volumes of the tetrahedra.

Lemma (Rivin, 1994)

$\mathcal{V}$ is strictly concave on a ideal hyperbolic tetrahedron.

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$\mathcal{V}$ is strictly concave on a ideal hyperbolic tetrahedron.

- Let p_1 and p_2 the smallest angles of a tetrahedron T.
- Let w a linear transformation over the angles of T, with coefficients w_1, w_2 and w_3 such that $w_1 + w_2 + w_3 = 0$.

$$-\frac{\partial^2 \mathcal{V}(T)}{\partial w^2} = \frac{(w_1 + w_2)^2 + (w_1 \cot p_1 - w_2 \cot p_2)^2}{\cot p_1 + \cot p_2}$$

The behavior of the volume

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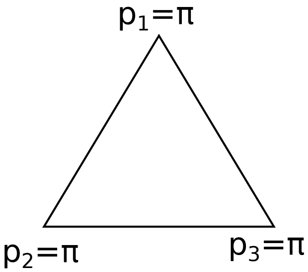

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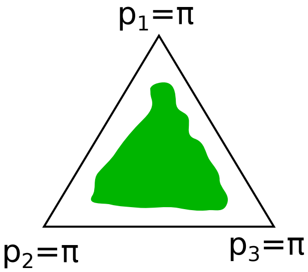

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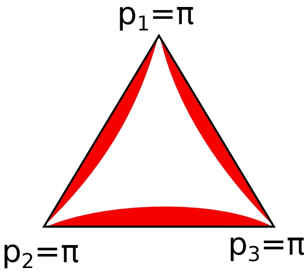
- If $\min(p_1, p_2)$ is constant, the derivatives are bounded by constants.
- If $\min(p_1, p_2) = x$ and $\max(p_1, p_2)$ is constant, then $\frac{\partial^2 \mathcal{V}(T)}{\partial w^2} = O_{x \to 0}(\frac{1}{x})$.



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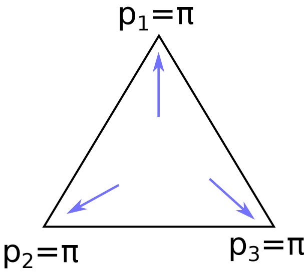
- If $(p_1, p_2) \to (0, 0)$...



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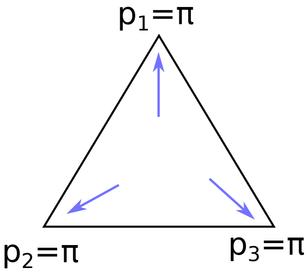
- If $(p_1, p_2) \to (0, 0)$... possible optimal on the boundary.



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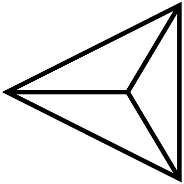

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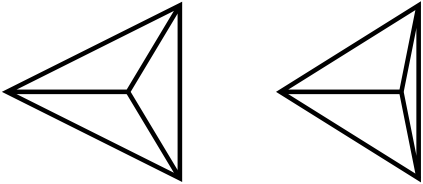

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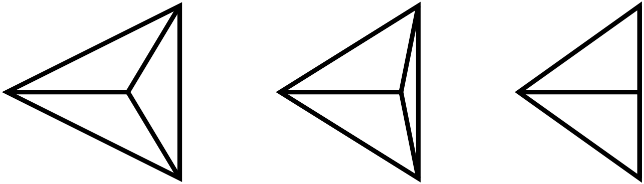
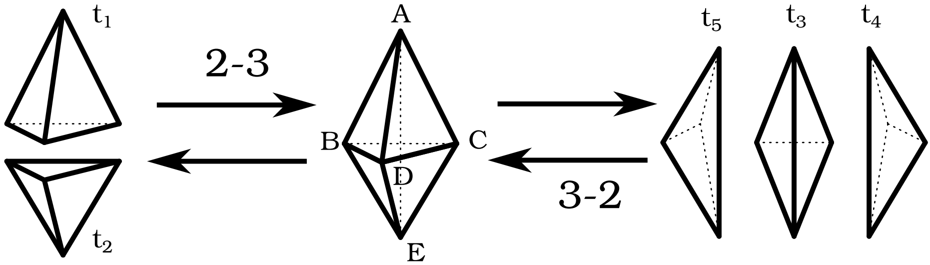


While (not geometric)
    maximize volume
    delete flat tetrahedra

Theorem (Pachner,1991)

Any two triangulations of a 3-manifold can be linked by a sequence of Pachner moves.



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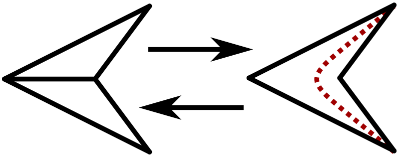
A 3-2 move is always geometric, a 2-3 is geometric iff the "external" angles are larger than π.



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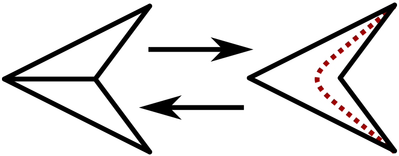

Remark

3-2 moves do not preserve the volume.

Flat tetrahedra deletion

Sequence of 2-3 moves followed by a 3-2.

Flat tetrahedra deletion

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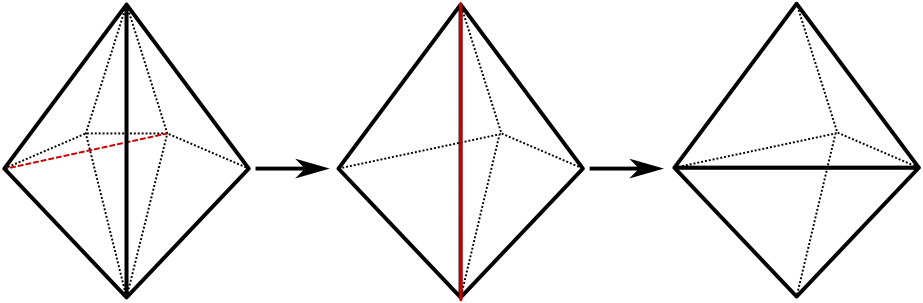

Recursive flat tetrahedra deletion

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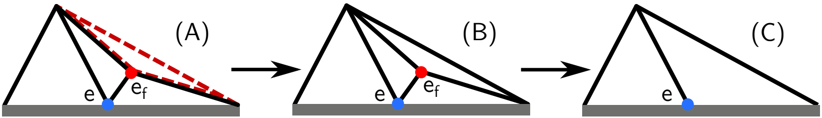
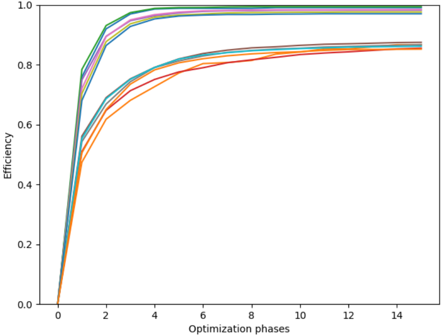
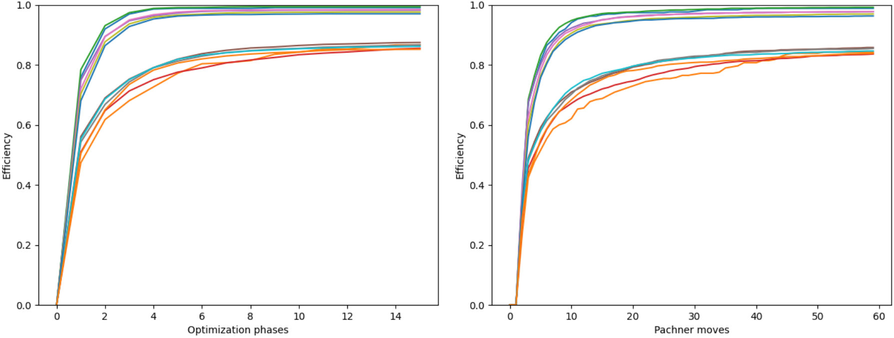


In situation (A) we want to create the red tetrahedron to be in situation (B), where a 3-2 can be applied.



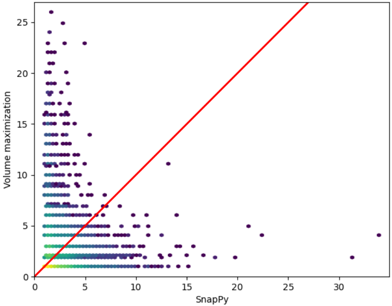
Right: Number of optimization phases required to find a complete hyperbolic structures.

Results



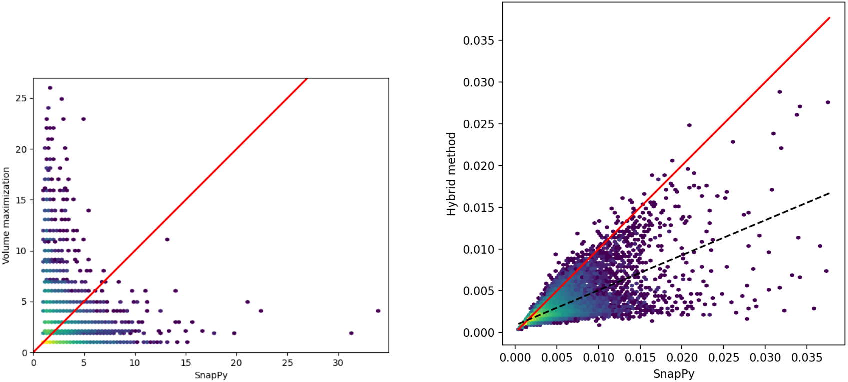
Right: Number of optimization phases required to find a complete hyperbolic structures. Left: Number of Pachner moves required to find a complete hyperbolic structures.

Results 2



Right: comparison of the difficult cases with SnapPy.

Results 2



Right: comparison of the difficult cases with SnapPy. Left: Time required to compute a complete hyperbolic structure in seconds, SnapPy compared to hybrid method.

Conclusion

Starting point

- To find a complete hyperbolic structure $\rightarrow$ solve gluing equations;
- not all triangulations admits a solution;
- there is a complete hyperbolic structure iff the hyperbolic volume is maximal.

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- To find a complete hyperbolic structure $\rightarrow$ solve gluing equations;
- not all triangulations admits a solution;
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Method

- Maximizing the hyperbolic volume leads to flat tetrahedra;
- these can be deleted to resume the maximization.

Results

- A lot of triangulations need few moves to accept complete hyperbolic structures;
- our method alone can be costly and not succeed;
- allows to improve on random re-triangulations when mixed.