The conjugacy problem in mapping class groups

Richard Webb
joint with Mark Bell

University of Manchester

July 19, 2022

Engineering and Physical Sciences Research Council

Some brief motivation

Markov 1935: knots or links in S^3 equivalent iff any pair of braid closure representatives differ by stabilisations and conjugacies

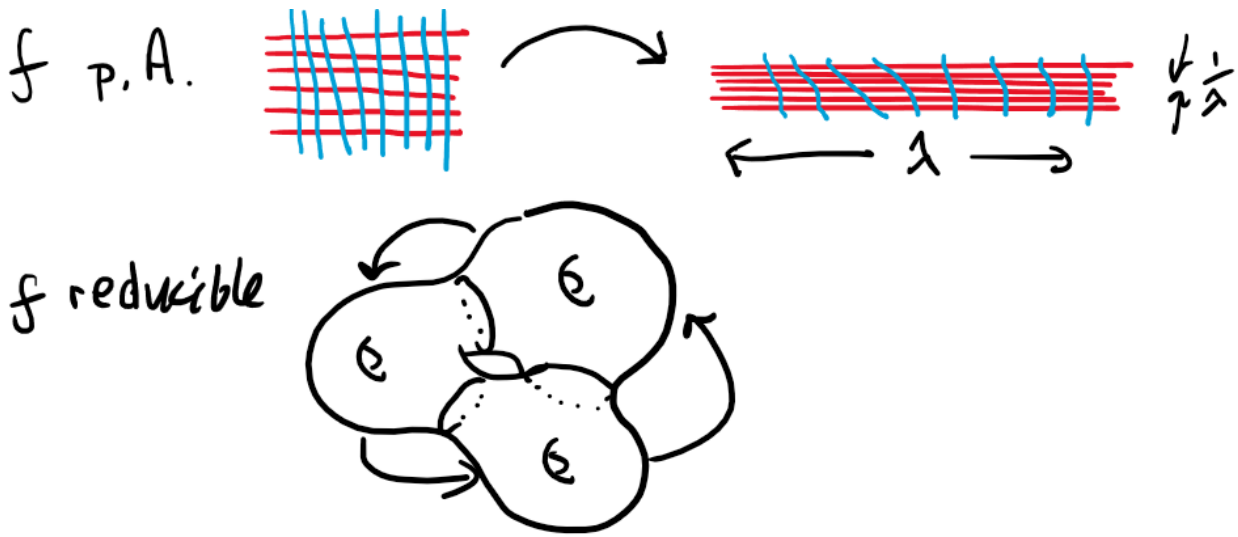
more generally: for S surface of finite type
$\mathrm{Mcg}(S)$ is the group of homeomorphisms $S \to S$ modulo isotopy

Hemion 1979 (pre-Thurston): conj. problem in $\mathrm{Mcg}(S)$ decidable
- important in Haken's algorithm

Important theorem

Theorem (Thurston 1979)

Let $f \in \mathrm{Mcg}(S)$ be of infinite order. Then either f is homotopic to a pseudo-Anosov map, or, preserves a system of essential curves.



General case: canonical reducing curve system, first return maps on subsurfaces, twist numbers, and so on. This info is needed for conjugacy, and we do this.

Note: Mosher tells us in quadratic time whether f has finite order.

Acknowledging previous work

Work by many authors, so this isn't an exhaustive list, or ordered correctly.

Various exponential time progress: Mosher, Hamidi-Tehrani and Chen, Koberda–Mangahas

Bestvina–Handel algorithm

Benardete–Gutierrez–Nitecki B_n, Calvez–Wiest cubic time algorithm for B_4

NP: for pA case Masur–Minsky, for general case J. Tao

co-NP: M. Bell

Main theorem

Theorem (in progress, Bell – W.)

Fix orientable finite type S and a finite generating set for $\mathrm{Mcg}(S)$*. Then the conjugacy problem lies in* P*.*

i.e. runs in polynomial time as a function of the word lengths of the input

also note

Theorem (in progress, Margalit–Strenner–Taylor–Yurttas)

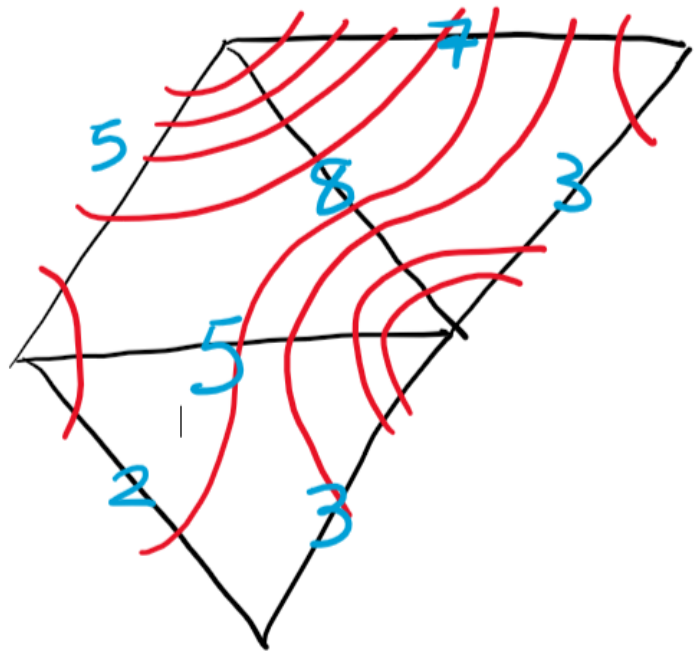
Fix orientable finite type S and a finite generating set for $\mathrm{Mcg}(S)$*. Then in quadratic time find the correct matrix to compute dilatation / stable lamination for pseudo-Anosovs / reducing curves.*

Both are in progress. Both rely on Nielsen–Thurston classification, and, simple closed curves "converging to" stable laminations quickly under iteration (coarse Hausdorff topology).

Our theorem is more geometric group theoretic; their theorem is more coordinate system oriented using $\mathcal{ML}(S)$.

Computational setup

We consider the action on (simple) curves on S, equipped with a triangulation $\mathcal{T}$. Have normal coordinates system



capturing intersections with each edge of $\mathcal{T}$. Arcs and other triangulations are similarly parametrised.
So $\mathcal{T}$ and a curve c gives a vector, whereas $\mathcal{T}$ and $f\mathcal{T}$ would give a matrix.

Simplified version of our strategy

Idea in the pseudo-Anosov case:
Conjugating a matrix corresponds to changing basis. Analogously, conjugating $f \in \mathrm{Mcg}(S)$ corresponds to changing the base triangulation $\mathcal{T}$.
What we do is, given f, compute only polynomially many triangulations $\mathcal{T}'$ then compute the matrices given by $\mathcal{T}'$ and $f\mathcal{T}'$.

These "canonical" triangulations $\mathcal{T}'$ in turn are determined by pairs of simple curves that *fill* S i.e. cut it up into discs/once-punctured discs.

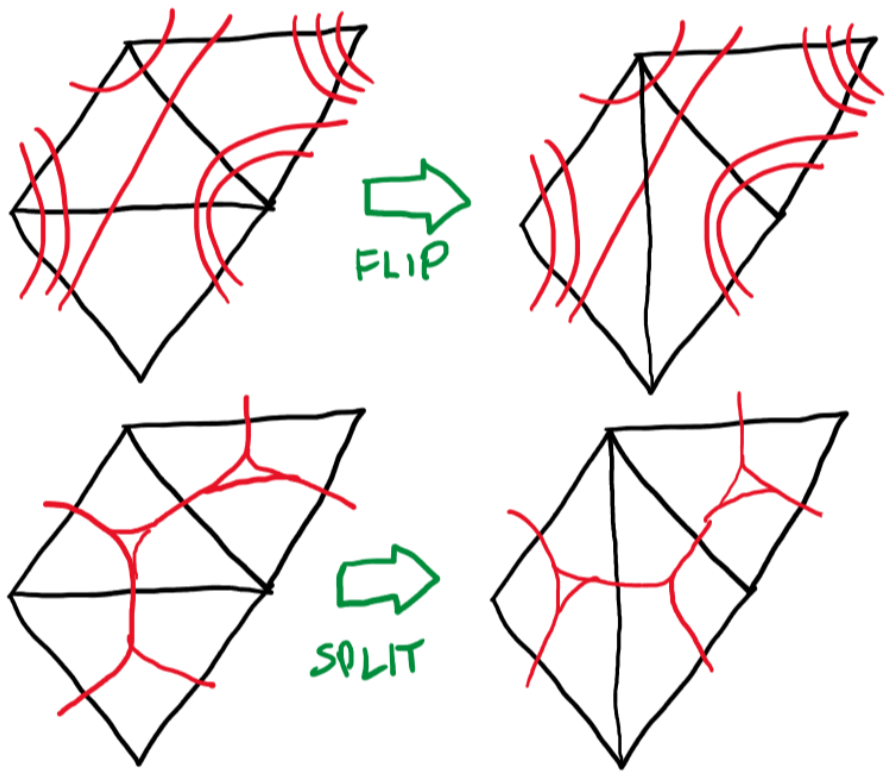
Theorem (Bell – W.)

Given a and b curves on $\mathcal{T}$, we can find/draw their minimally intersecting representatives, compute essential curves in $\partial n(a \cup b)$ if they do not fill S, and compute the "canonical" triangulations above if they fill S, all in polynomial time.

This seems to be new. Note: Schaefer–Sedgwick–Stefankovic already proved geometric intersection number can be computed in poly. time, using different methods.

Brief idea

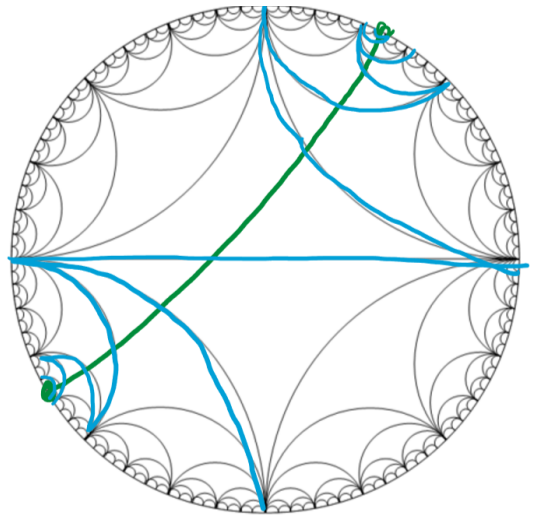
Change the triangulations, giving a sequence $\mathcal{T}_n$, until one of the curves has very small intersection with $\mathcal{T}_n$. Then you're almost done. How to flip:



Problem: splitting can get stuck spiralling around one curve! Very bad! This is overcome by using ideas of Agol–Hass–Thurston. Our implementation is more similar to Erickson–Nayyeri.

The truth

We compute enough pieces of the "axis" for a pseudo-Anosov f on the curve graph $\mathcal{C}(S)$. The canonical triangulations then come from filling pairs of curves along this axis.



For the infinite-order reducible case, we extract a reducing curve system because a large power f^N of f will "rotate" around a reducing curve system, which can be picked out.

The details of the above use some curve graph machinery, such as tight geodesics / tight paths. So as a bonus, we can compute geodesics between vertices in the curve graph in polynomial time!

Finite order case

Very briefly described!

The finite order case is quite different to the infinite one. For punctured S we can do it in quadratic time and poly. in S. It's implemented in Mark Bell's Curver.

For punctured S, we use "unicorn arcs", Hatcher flow / Mosher flip sequences in order to find invariant arcs / curves under f. Repeat this until you cut S up into small pieces.

Then you obtain the quotient orbifold $S/\langle f \rangle$, and the deck transformation determined by f. This is good enough for the conj. problem - can also do conj. for finite subgroups of $\mathrm{Mcg}(S)$ similarly.

For closed S it is harder, since we don't have arcs, but a similar idea works using "bicorn curves".

Thank you!