Computational Complexity and Knot Theory

Arnaud de Mesmay (CNRS, LIGM, Université Gustave Eiffel, Paris)

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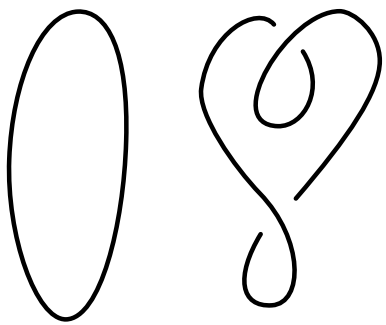
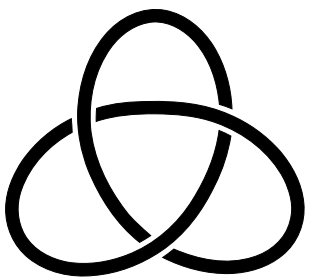
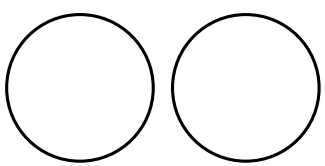
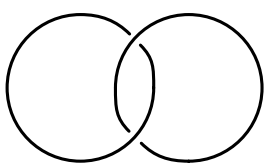
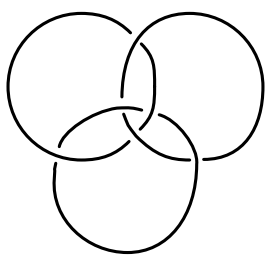
Knot Theory

Knots

- A *knot* is a nice map $K : S^1 \to \mathbb{R}^3$.
- Two knots are *equivalent* if they are *ambient isotopic*, i.e., if there exists a continuous deformation from one into the other without crossings.
- A *link* is a disjoint union of knots.

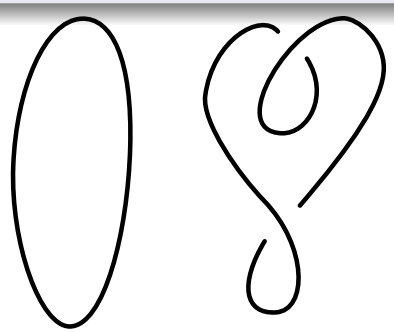
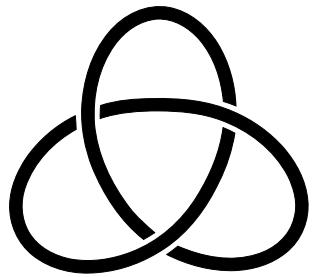



Knot diagrams

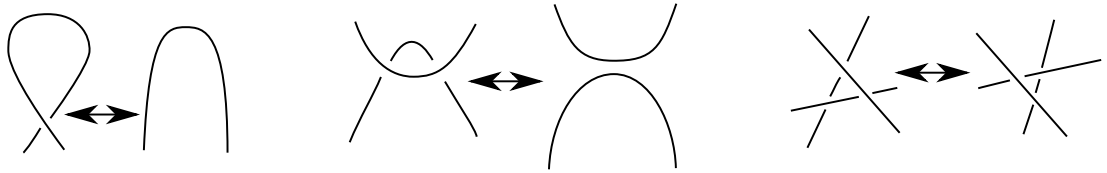
Diagrams

- A *knot diagram* is a 2D-projection of a knot where at every vertex, one indicates which strand goes above and below.
- The *crossing number* of a knot K is a minimum number of crossings over all knot diagrams for K.


Theorem (Reidemeister)

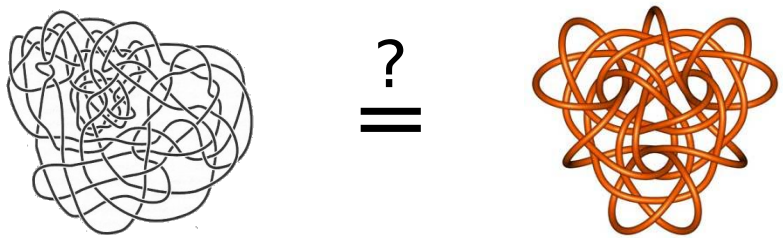
Two knot diagrams correspond to equivalent knots if and only if they can be related by a sequence of Reidemeister moves.



Knot equivalence

Knot equivalence

Input: Two knots K_1 and K_2 represented by diagrams.
Output: Is K_1 equivalent to K_2?



What is the best known algorithm for this problem?

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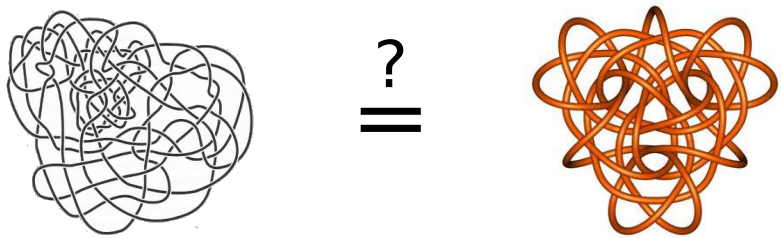

What is the best known algorithm for this problem?

- Decidable [Haken'68, Hemion '79, Matveev '07].
- Best bound on Reidemeister moves is from [Lackenby-Coward '14]:

$$2^{2^{\cdot^{\cdot^{2^{n_1+n_2}}}}} \Big\} \text{ height } c^{n_1+n_2} \text{ where } c = 10^{1000000}.$$

- [Kuperberg '19] provides an elementary algorithm, i.e., with a tower of exponentials of constant size.

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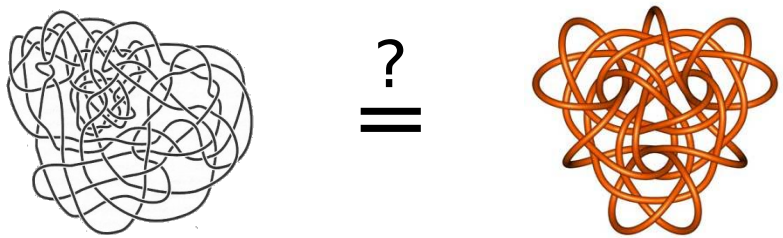

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This problem does not look easy. Perhaps one can prove that it is hard?

The very basics of computational complexity I

- A decision problem is in **P** (polynomial-time) if there exists an algorithm solving instances of size n in time $p(n)$, where p is a polynomial.
- A decision problem is in **NP** (non-deterministic polynomial-time) if there exists an algorithm *verifying* positive instances of size n *with a hint of size* $h(n)$ in time $p(n)$, where p and h are polynomial.

Example: Non-primality

One can easily verify that a number is non-prime when one is given its prime factors as a hint.

- A decision problem is in co-**NP** if its complement is in **NP**.

Example: Primality

Primality testing is in co-**NP** because of the previous example.

The very basics of computational complexity II

Standard conjectures

P $\neq$ **NP** $\neq$ co-**NP**.

- A problem is **NP**-hard if any problem in **NP** reduces in polynomial time to it. In particular, a problem being both in **P** (resp. co-**NP**) and **NP**-hard would mean that **P** = **NP** (resp. co-**NP**=**NP**).
- Rule of thumb in many parts of theoretical computer science (e.g., graph theory): every reasonable problem, except a few well-known exceptions, is in **P** or is **NP**-hard (see for example the Feder-Vardi conjecture [Bulatov, Zhuk '17])
- Most problems in knot theory are not known to fit in this dichotomy.

NP-hard problems in knot theory

- Is knot equivalence **NP**-hard?

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- Is knot equivalence **NP**-hard? We do not know.
- It is consistent with the state of the art that it can be solved in linear time.

NP-hard problems in knot theory

- Is knot equivalence **NP**-hard? We do not know.
- It is consistent with the state of the art that it can be solved in linear time.
- Most problems in knot theory are very hard to compute in practice. Is any of them **NP**-hard?
- When I started working on these problems ten years ago, there were to my knowledge only two known computational hardness results in knot theory:
  1. Given a knot K in a 3-manifold M, is the genus of K at most g? [Agol-Hass-Thurston '95]
  2. It is $\#P$-hard to compute [Jaeger, Vertigan, Welsh '90], or even approximate [Kuperberg '15] the Jones polynomial of a knot.

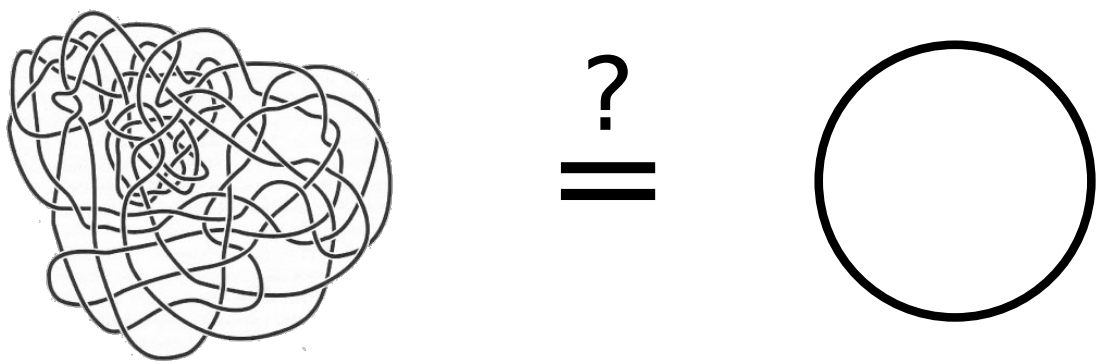
Since then, many new results, but still many open problems.

Why should we care?

1. **NP**-hardness traces a line between problems that can be solved in polynomial time and those that cannot.
2. Some problems in knot theory are very likely not **NP**-hard:

Unknot recognition

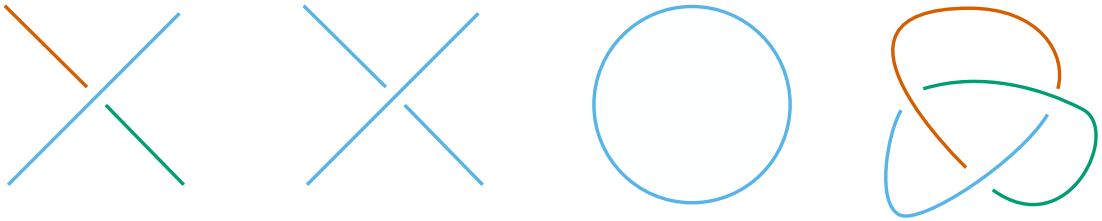
Input: A knot K represented by a diagram.
Output: Is K equivalent to the trivial knot?



- In $\mathbf{NP} \cap co - \mathbf{NP}$ [Hass-Lagarias-Pippenger '99], [Agol'02 $\rightarrow$ Lackenby '18].
- Actually, computing the genus is in $\mathbf{NP} \cap co - \mathbf{NP}$ [Lackenby '21], even in a fixed 3-manifold [Lackenby-Yazdi '20]

Some knot invariants

- Tri-colorability: Can I color my knot using three different colors and the following rules?



- More generally: does there exist a non-trivial homomorphism of $\pi_1(\mathbb{S}^3 \setminus K)$ into a fixed finite group G that sends a meridian to a fixed conjugacy class c?

Theorem (Kuperberg-Samperton '21)

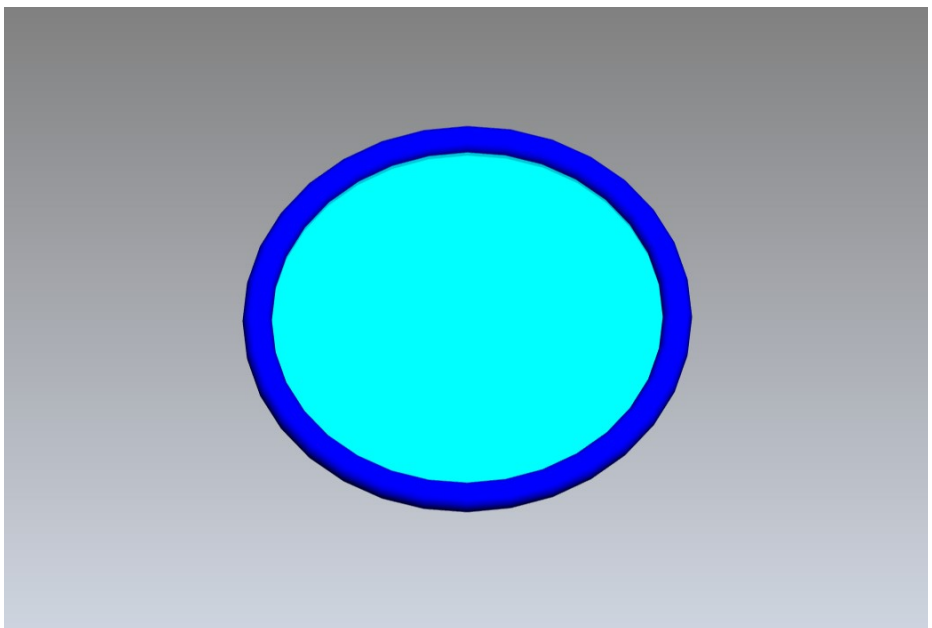
*If G is fixed, non-abelian and simple (for example $\mathcal{A}_5$), this problem is **NP**-hard. (More generally counting the number of representations is $\#P$-hard.)*

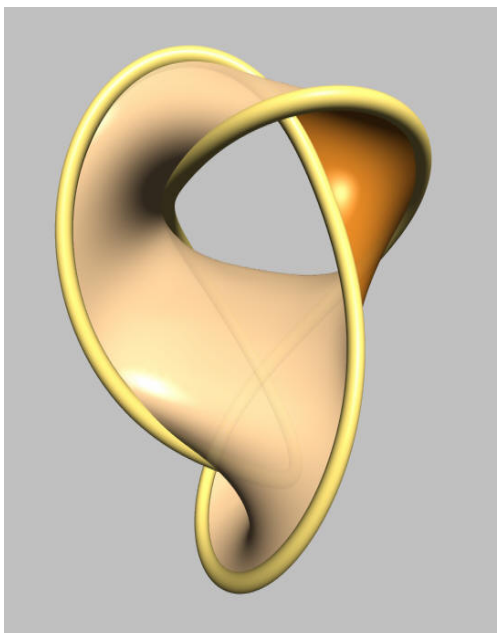
- Polynomial invariants: Computing the Alexander polynomial of a knot can be done in *polynomial-time*, but computing/approximating the Jones polynomial is $\#P$-hard [Jaeger-Vertigan-Welsh '90, Kuperberg '15].

These are the only* known hard problems for *classical knots*. Other hardness results work with *knots in 3-manifolds*, or with *links*.

Some previous NP-hardness results

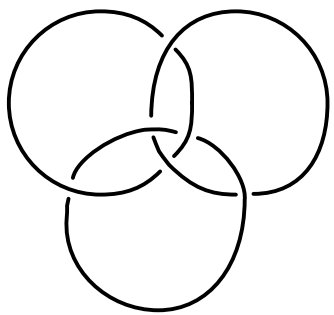
- Knot genus in a 3-manifold [Agol-Hass-Thurston '05].
- Thurston norm of a link [Lackenby '17].





Some previous NP-hardness results

In [dM-Rieck-Sedgwick-Tancer '21], we started from a nice similarity between Borromean rings and SAT clauses to show that the following problems are NP-hard:

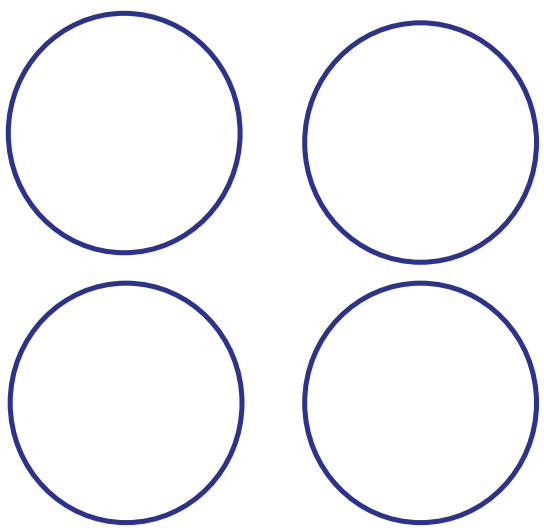


$$C = v_1 \vee v_2 \vee v_3$$

Some previous NP-hardness results

- Finding a sublink [Lackenby '17], finding a trivial sublink [Koenig-Tsvietkova'21],[dM-Rieck-Sedgwick-Tancer '21].

$$\phi = (t \vee x \vee y) \wedge (\neg x \vee y \vee z)$$



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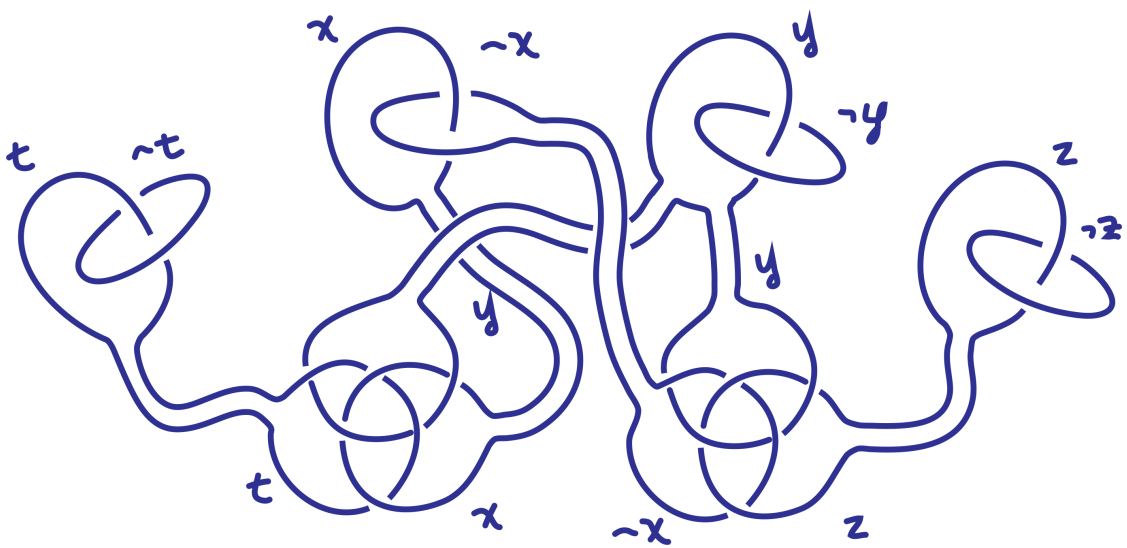
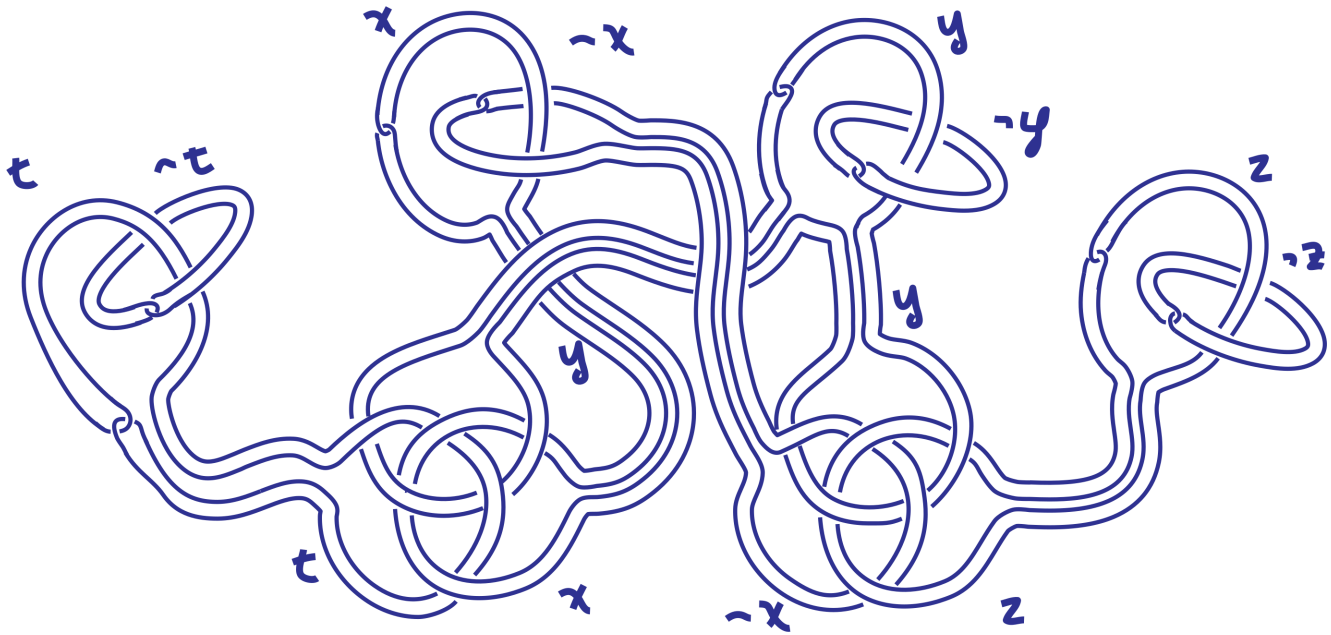


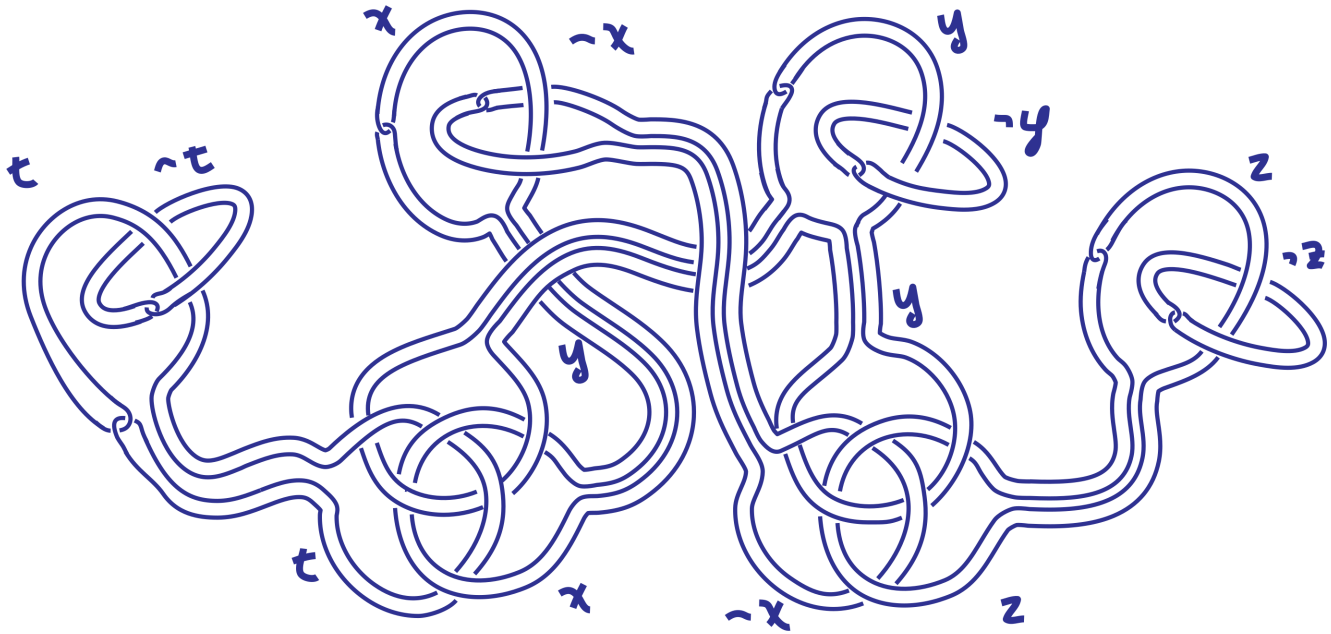
Some previous NP-hardness results

- Unlinking number [Koenig-Tsvietkova'21],[dM-Rieck-Sedgwick-Tancer'21].



Some previous NP-hardness results

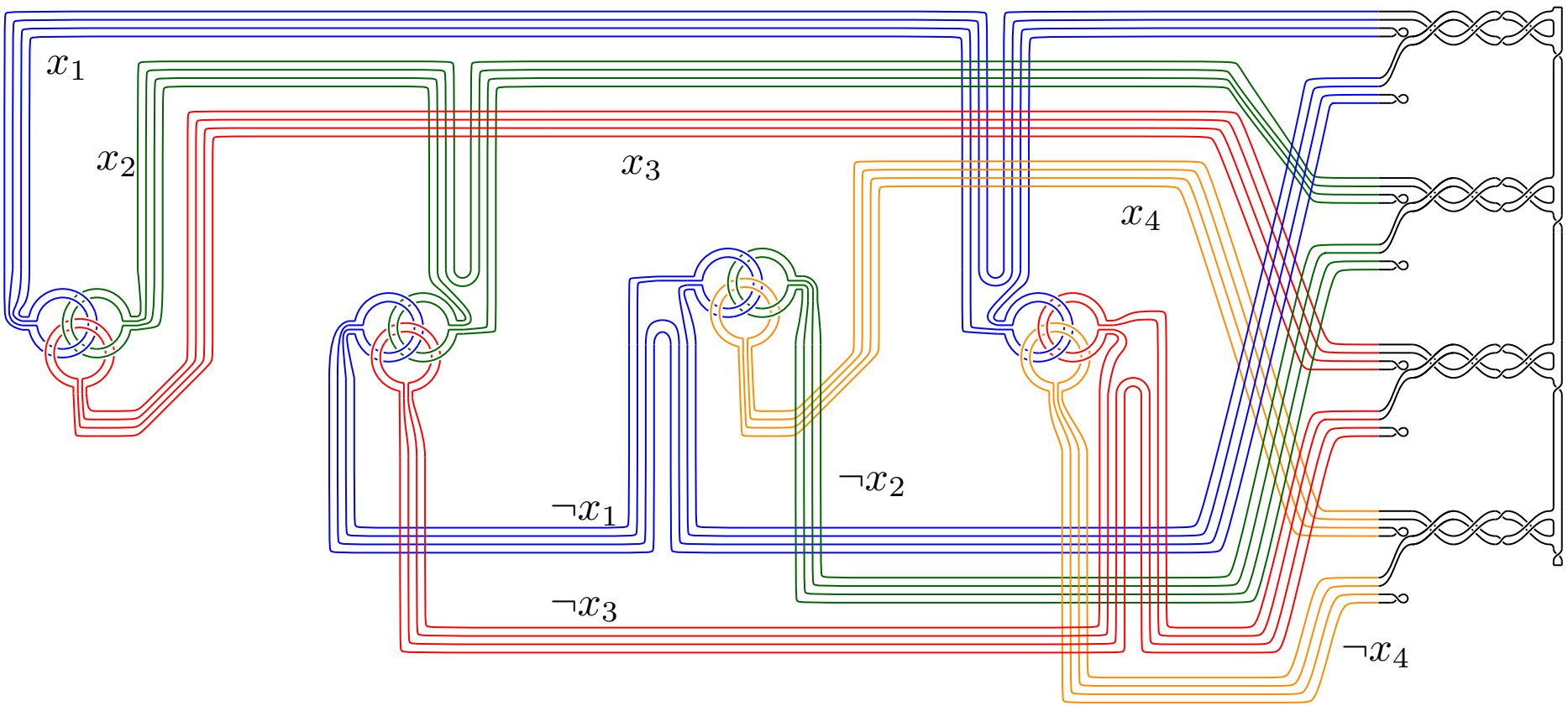
- Four-ball Euler characteristic $\chi_4(L)$ [dM-Rieck-Sedgwick-Tancer'21].



Some previous NP-hardness results

- Deciding whether a *knot* can be turned into a trivial diagram using at most k Reidemeister moves [dM-Rieck-Sedgwick-Tancer '21].

$$\Phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_4) \wedge (x_1 \vee \neg x_3 \vee \neg x_4)$$



Computing the crossing number

Crossing number

Input: A knot/link diagram D.
Output: Is the crossing number of the knot/link at most k?

Best known algorithm to decide whether the crossing number of a link L is at most k:

for $i = 1 \ldots k$ **do**
    for All the link diagrams D with i crossings **do**
        Test whether D and L are the same link.
    end for
end for

Marc Lackenby

"[This algorithm] is obviously not very efficient but it seems unlikely that there is any quicker way of determining a link's crossing number in general."

Hardness of the crossing number

Theorem (Schaefer, Sedgwick, dM'20)

*The crossing number problem for links is **NP**-hard.*

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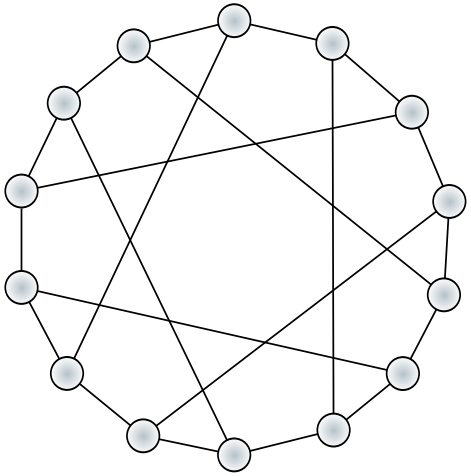
Theorem (Schaefer, Sedgwick, dM'20)

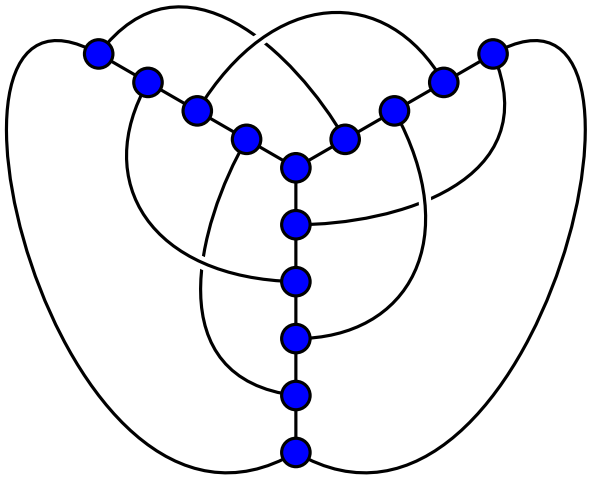
*The crossing number problem for links is **NP**-hard.*

- First reaction: of course it's **NP**-hard, there should be an easy reduction from the *graph crossing number*.
- Yes but not that easy: it is open whether the crossing number of a *knot* is **NP**-hard.

A naive reduction

- The *crossing number* of a graph G is the minimum number of edge crossings in a plane drawing of G.





- It is notoriously unwieldy, for example the exact values of the crossing numbers of complete graphs K_n and complete bipartite graph $K_{m,n}$ are unknown.

Theorem (Garey-Johnson '83)

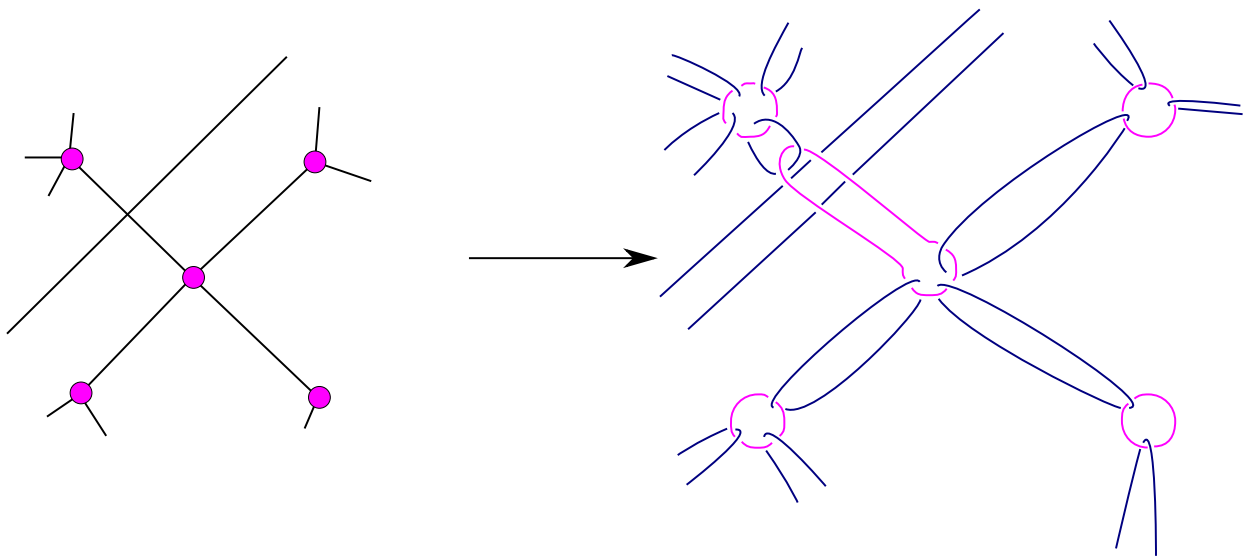
Computing the crossing number of a graph is ***NP****-hard.*

A naive reduction

Theorem (Garey-Johnson '83)

*Computing the crossing number of a graph is **NP**-hard.*

- Transforming a graph into a link.



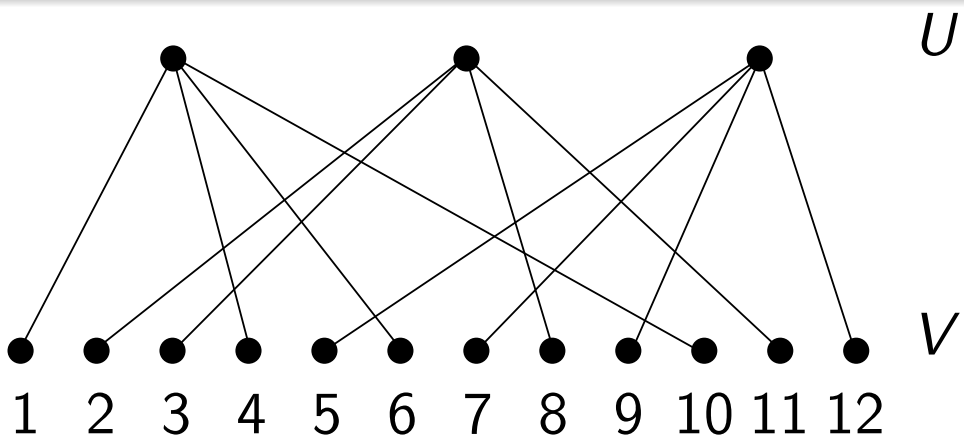
- But when changing the cyclic ordering around the vertex, the link gets all tangled up.
- We must prevent the components corresponding to vertices from stretching.

The actual reduction

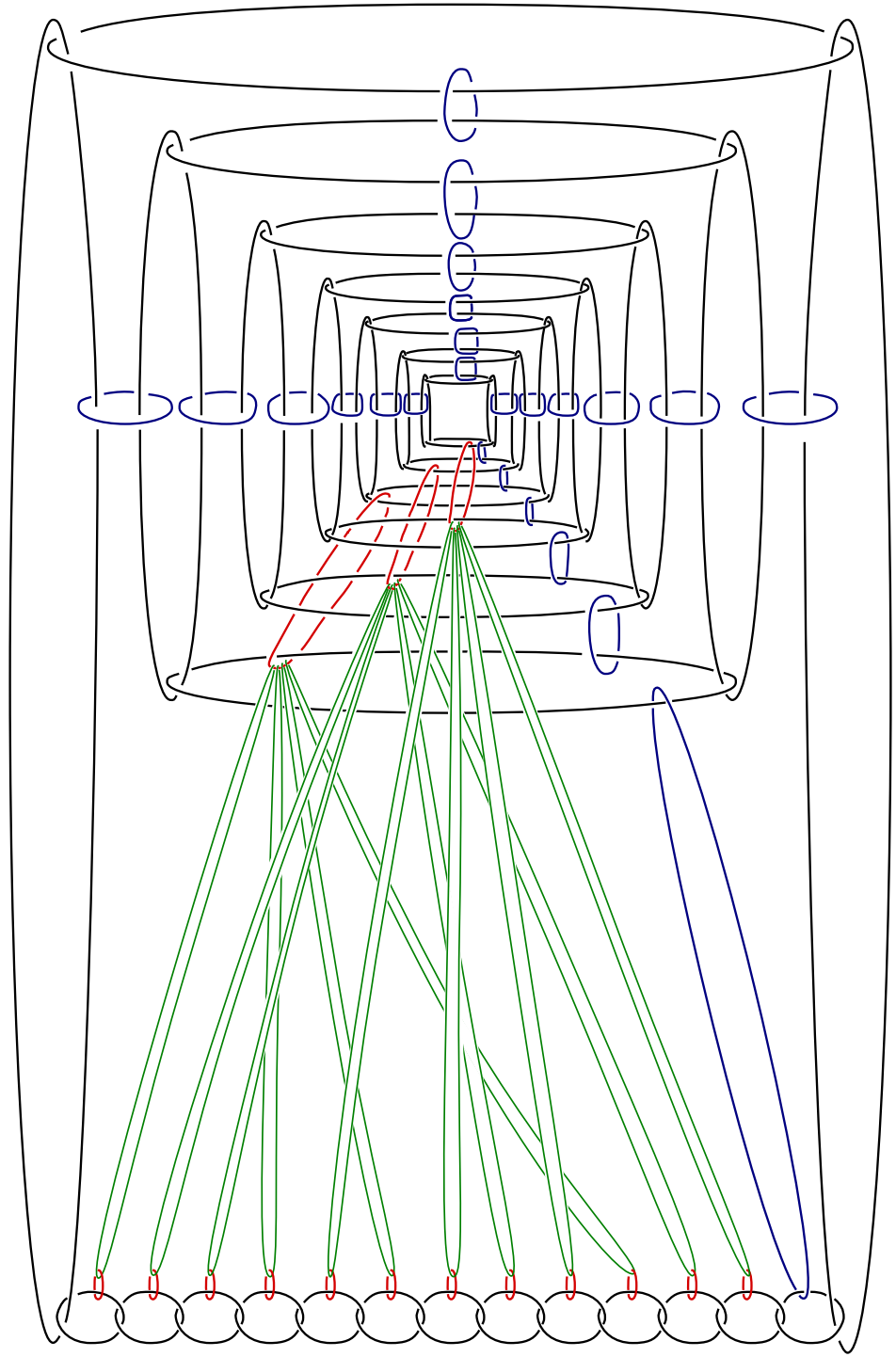
We reduce from a specific variant of the graph crossing number, where the cyclic orderings are fixed:

Theorem (Muñoz-Unger-Vrťo '02)

*Determining the bipartite crossing number of a bipartite graph $G = (U \cup V, E)$ in which all vertices in U have degree 4, all vertices in V have degree 1, and the order of the V-vertices along their line is fixed, is **NP**-complete.*



which we transform into...

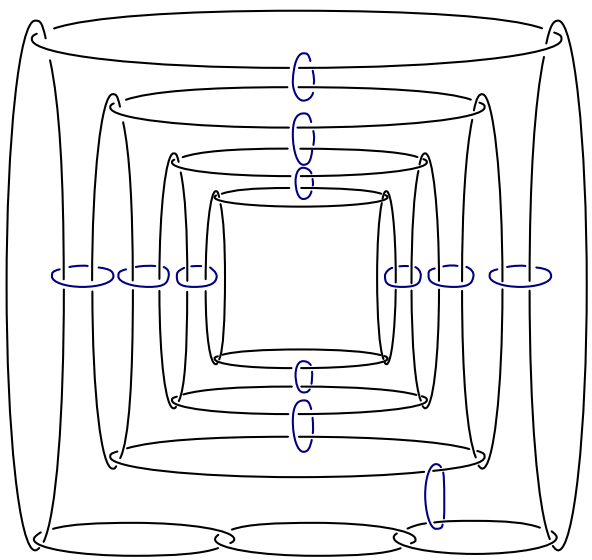


Why does this work?

One direction is immediate: from a graph drawing with low crossing number we get a link diagram with low crossing number.

For the other direction, we want to prove that in any diagram of low crossing number, things are as we would expect:

- the frame is rigid and

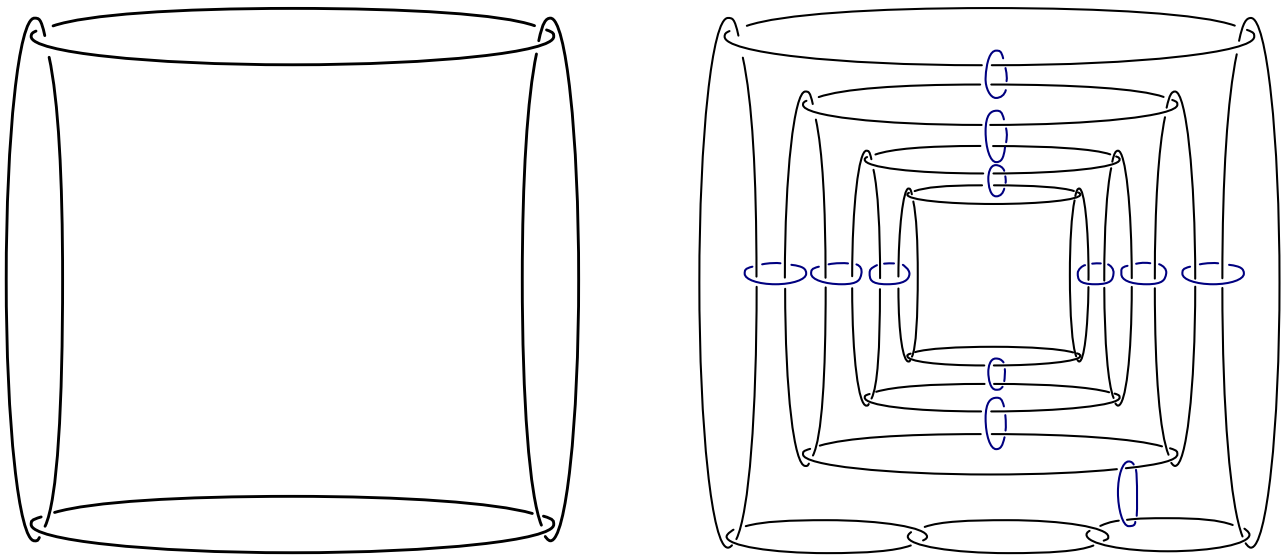


- the only things moving are the red curves.

Using linking numbers

Main tool: Linking numbers.

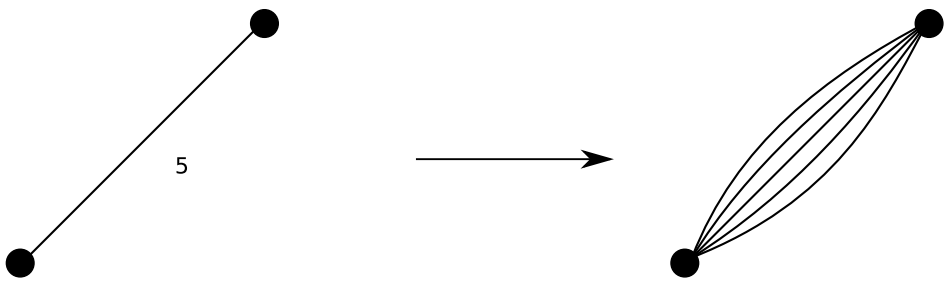
- With linking numbers, we can prove that this diagram of the frame is the unique one with a minimal number of crossings.



- Then the hope is that the placement of the frame forces other crossings (even those not forced by linking numbers).
- But adding the other gadgets may break the rigidity of the frame.

Weighted crossings

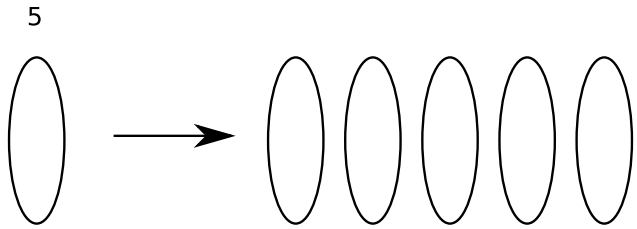
- This is a common issue in reductions involving crossing numbers.
- We can gain rigidity by putting big *weights*: each edges has a weight w_e, and the weighted crossing number of e and f crossing is $w_e w_f$.
- This can be easily simulated by using multiple edges.



- In the setting of graphs, it is immediate that all the multiple edges will be drawn the same way in some crossing-minimal drawing.
- Big weights can enforce rigidity.

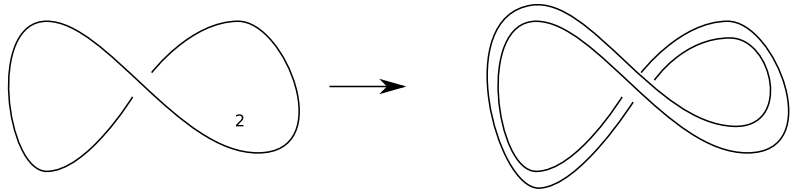
Weighted knots

Likewise, we can use multiple copies of knot to represent weights:

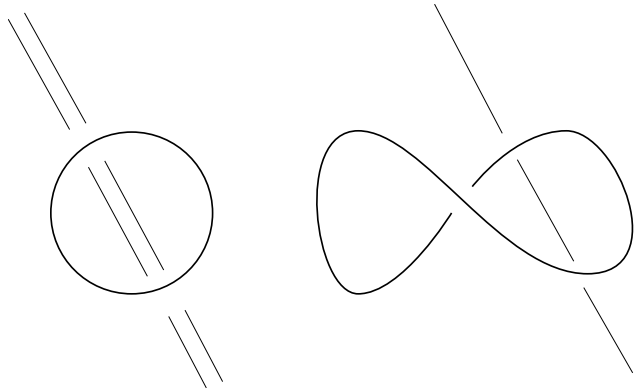


However:

- Self-crossings throw off the accounting, hence we use unknots in the reduction.



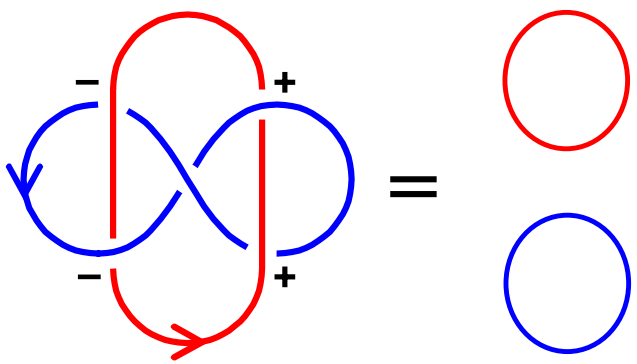
- We can not argue that in a crossing-minimal drawing, all the copies of a knot will be drawn the same way.



Our solution

- We do use weighted knots, and choose weights wisely.
- When arguing that things look like we want them to look, we use a relaxed notion of equivalence.

Two links are *parity-link equivalent* if the parity of the linking number between pairs of components is the same in both crossings.

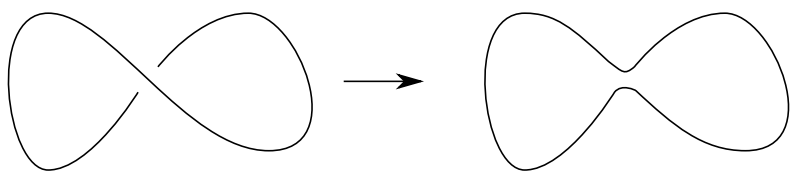


Parity-link equivalence is simpler to handle

Lemma

For any link L, let D' be a diagram with a minimum number of crossings of a link L' which is parity-link equivalent to L. Then no link component in D' has self-crossings.

Proof:



Working from a different link.

- The argument showing that the frame is rigid is only based on linking numbers!

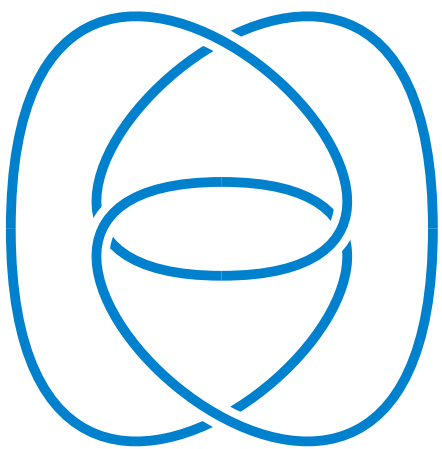
So, if L has a drawing with a low crossing number:

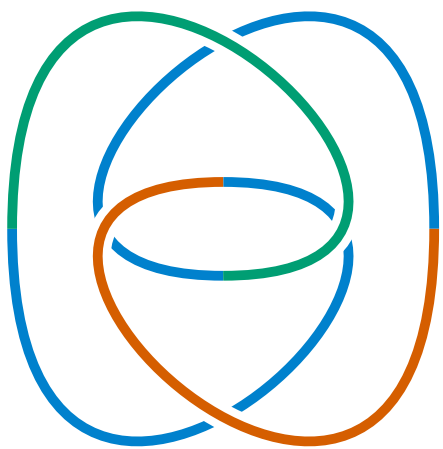
- We look at the crossing-minimal drawing D of a link L' that is parity-link equivalent to our link L. It also has a low number of crossings.
- L' might be different from L, but it does not matter:
- There, the frame is rigid.
- Likewise, the only non-rigid pieces are the moving red curves.
- We can find a drawing of our original bipartite graph from D with few crossings.

We also get **NP**-hardness for the minimal crossing number under other notions of equivalence: parity-link equivalence, linking-number equivalence, link-homotopy and link concordance.

What next?

- How to adapt this to knots? Or links with a bounded number of components? *Alternating knots* might help but the weighting issue is problematic.
- Is it still hard for a fixed value of the crossing number? Note that for a fixed k, determining whether a graph has crossing number at most k can be done in linear-time [Kawarabayashi-Reed'07].
- What about the *bridge number*? (minimum number of *bridges* to draw the knot)





One more speculative slide

What is the computational complexity of knot problems where one forces to stay in the PL category with at most k segments? For example:

- Complexity of the *stick number* (minimum number of segments to realize a knot)?
- Complexity of deciding whether two knots made of k segments can be isotoped to each other using knots made of k segments?

I would expect strong connections to the theory of *linkages* and the *existential theory of the reals*. In particular, is the space of realizations of a knot universal in the sense of Mnev?

Open problem: Stuck geometric unknots, [Calvo '01]

Are there topological unknots which cannot be untangled geometrically in the sense above?

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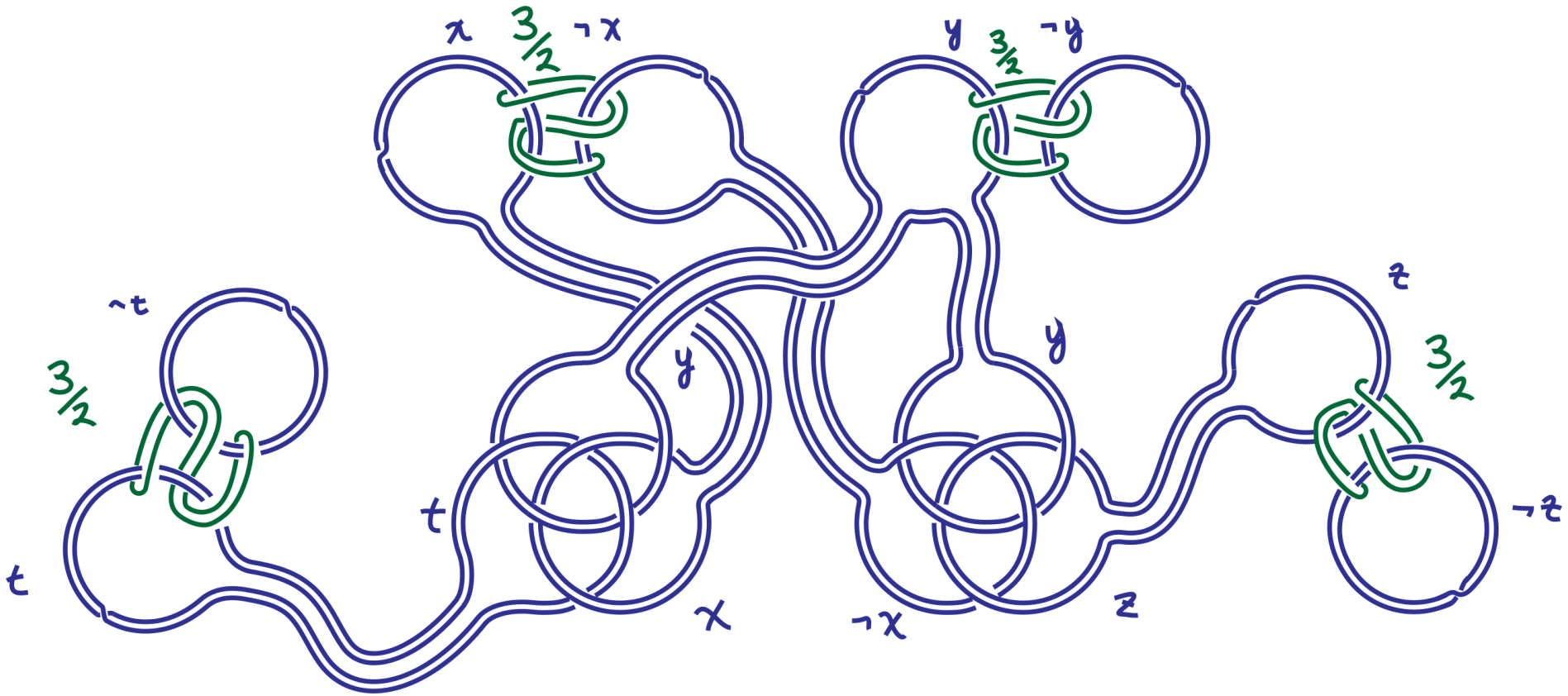
Thank you! Questions?

Our original motivation: $\textsc{Embed}_{2\to 3}$ and $\textsc{Embed}_{3\to 3}$



This 3-manifold embeds into $\mathbb{S}^3$ if and only if Φ is satisfiable.
$\to$ Deciding whether a 3 or a 2-dimensional space embeds into $\mathbb{R}^3$ is **NP**-hard.
Best algorithm runs in a tower of exponentials [Matoušek, Sedgwick, Tancer, Wagner '16].