Parameterized complexity in low dimensional topology

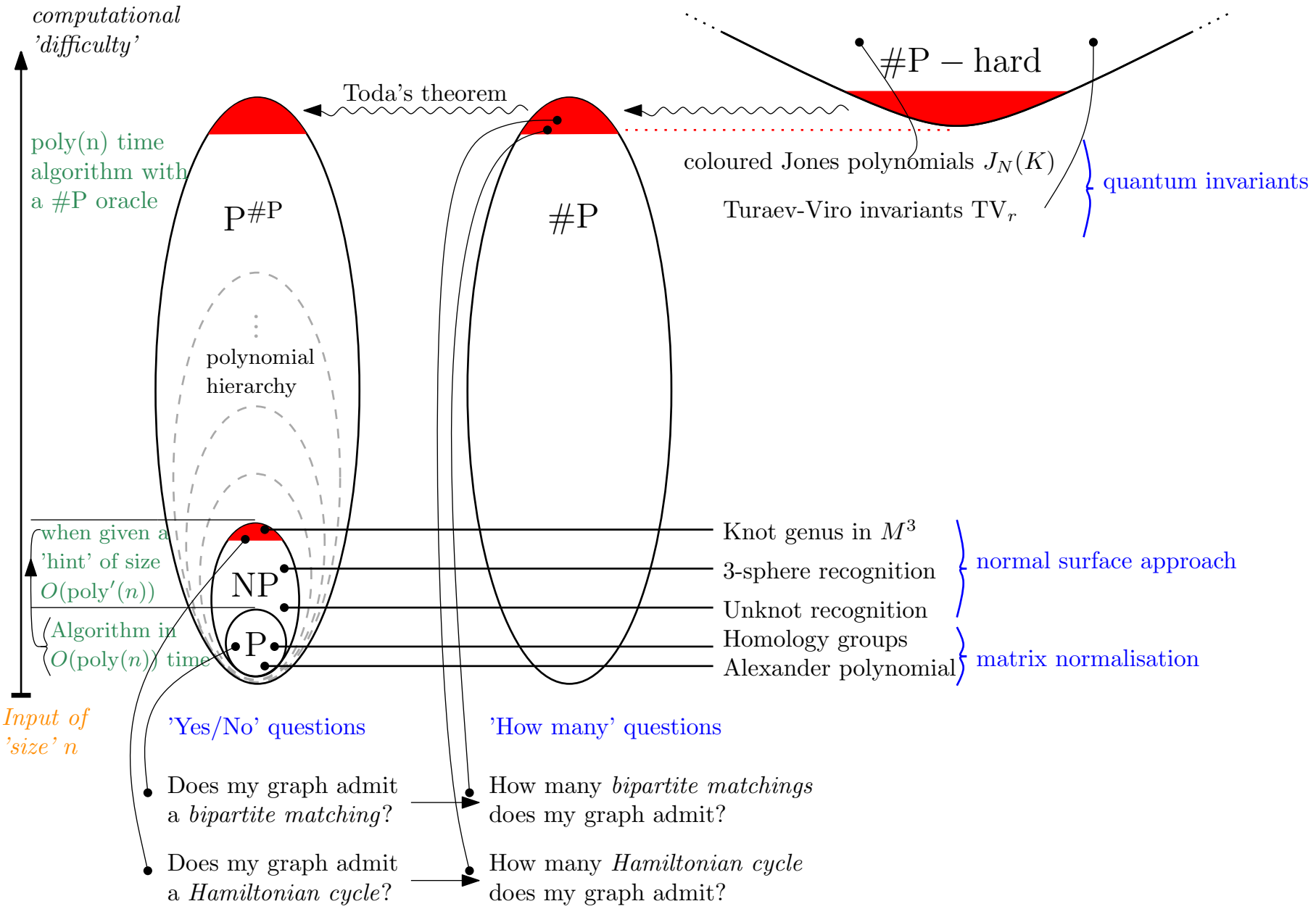
Clément Maria

INRIA Université Côte d'Azur, FRANCE

AMS-EMS-SMF Joint International Meeting

Some complexity

A map of complexity classes



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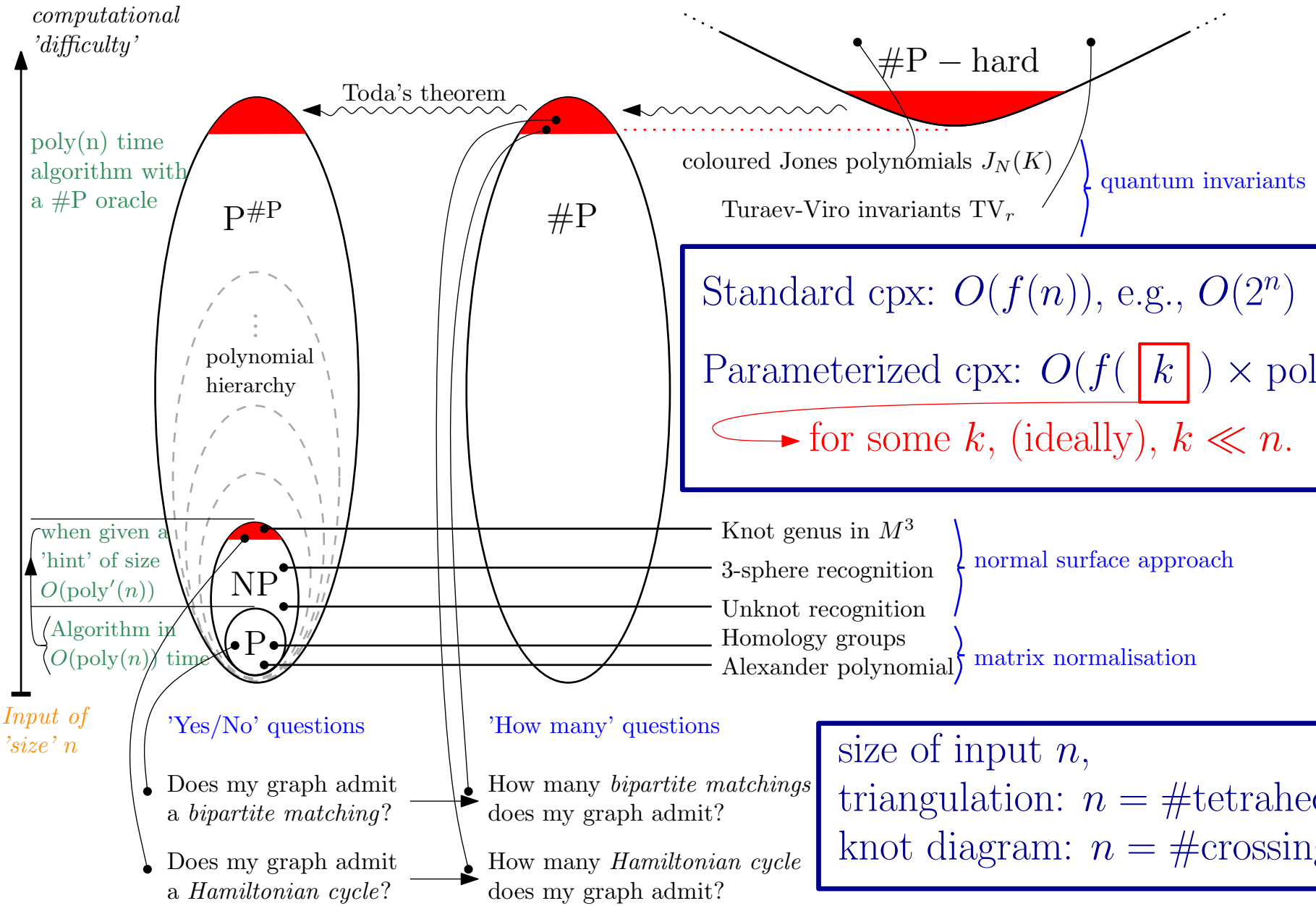

Standard cpx: $O(f(n))$, e.g., $O(2^n)$

Parameterized cpx: $O(f(\boxed{k}) \times \text{poly}(n))$

for some k, (ideally), $k \ll n$.

size of input n,
triangulation: $n = \#\text{tetrahedra}$
knot diagram: $n = \#\text{crossings}$

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Best scenario: k is of topological nature, e.g.,

Theorem (M., Spreer '16)

For any input triangulation T, of size n, of a manifold M, there is a

$$O(2^{\beta_1} \times n^3)$$

time algorithm to compute the Turaev-Viro invariant $TV_{4,1}(M)$ (which is #P-hard), where $\beta_1 = \dim H_1(M, \mathbb{Z}/2\mathbb{Z})$.

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- parameter independent of triangulation (the "presentation")
- the problem becomes P for all manifolds of bounded β_1
- unfortunately hard to generalise.

Combinatorial parameters

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More systematic: k is of combinatorial nature, e.g., k measures the sparsity of a triangulation/knot diagram called width.

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Theorem (M. '20)

For a knot diagram of a knot K, with n crossings and width w, the coloured Jones polynomial $J_N(K)$ (or RT-inv.) can be computed in:

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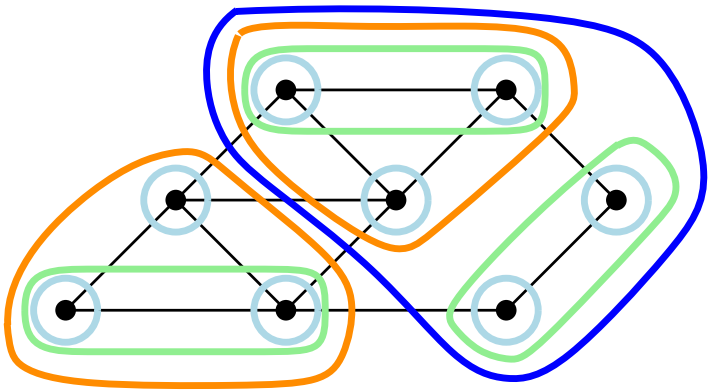
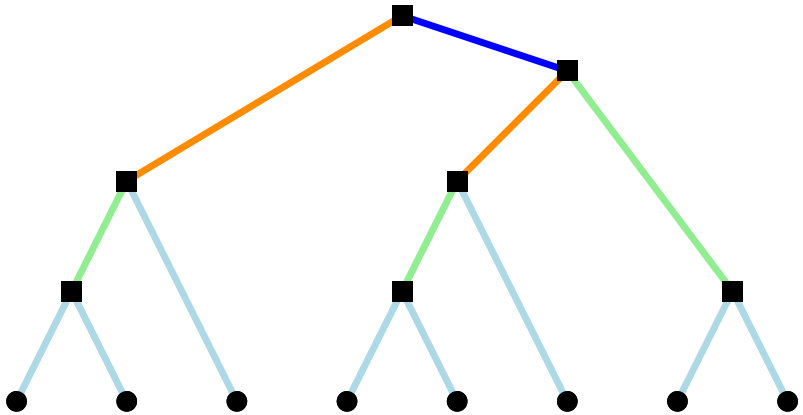
Carving-width of a graph

For planar graph (but can be defined on general graphs):





Graph vertices $\leftrightarrow$ Leaves of bin. tree | Tree edge $\Rightarrow$ graph cut

- System of Jordan curves, cutting the graph transversally,
- The width is the maximal number of intersections between a Jordan curve and the graph edges.

The smallest possible width is the carving-width cw.

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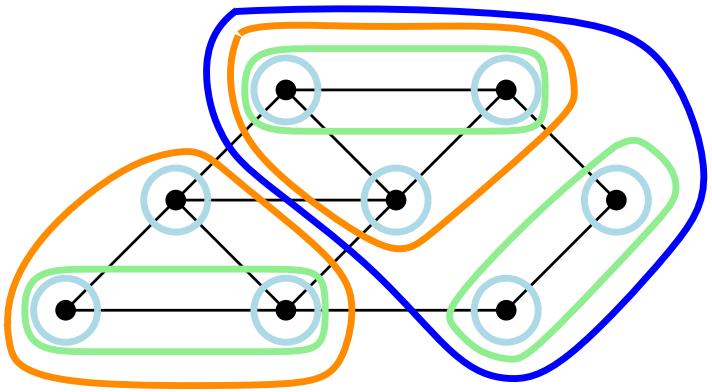

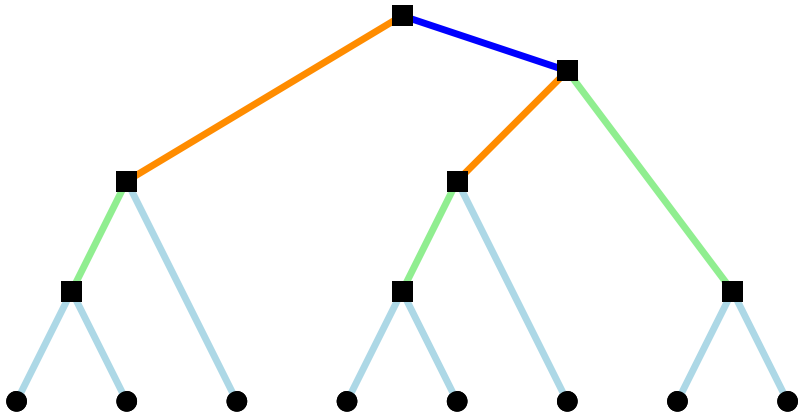

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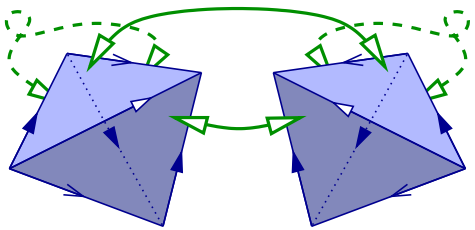
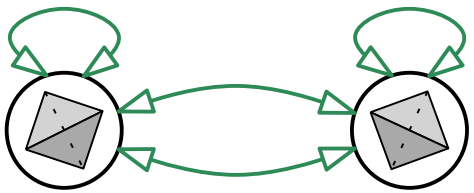
Very close to graph *tree-width*.

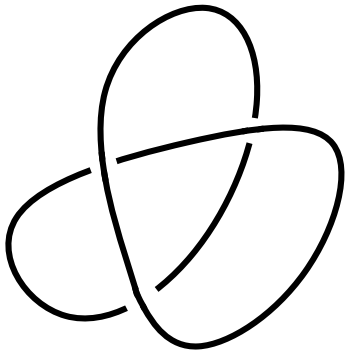
Carving-width of a manifold/knot

A triangulation induces a 4-valent graph:





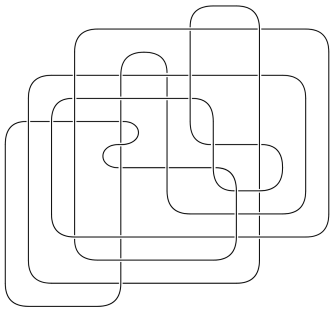
A knot diagram gives a 4-valent planar graph.



The (carving-)width $\mathrm{cw}(M)/\mathrm{cw}(K)$ of a manifold M/knot K is the minimal (carving-)width over all of its triangulations/diagrams.

A reasonable strategy

Any diagram of
a knot K,
- n crossings,
- arbitr. width.



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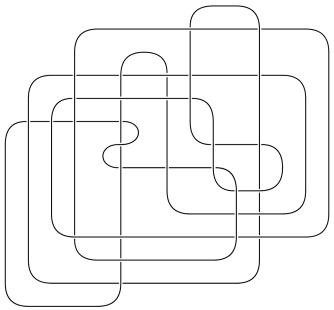

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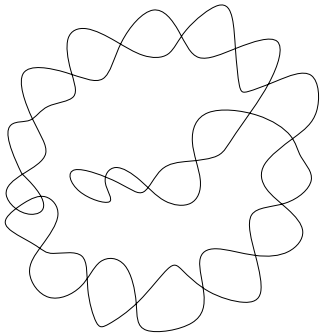

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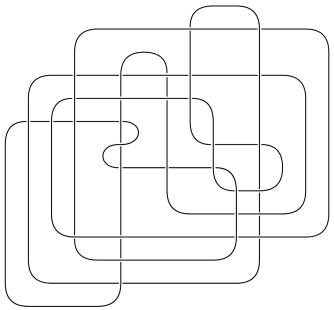

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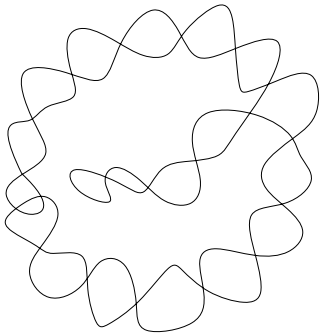

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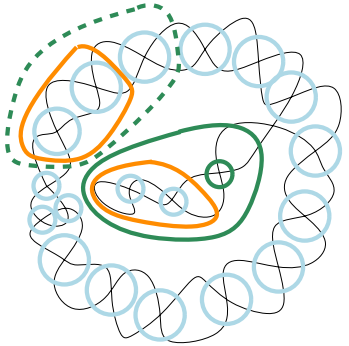
Computation of a (close to optimal) width-decompostion $g \circ f(\mathrm{cw}(K))$.



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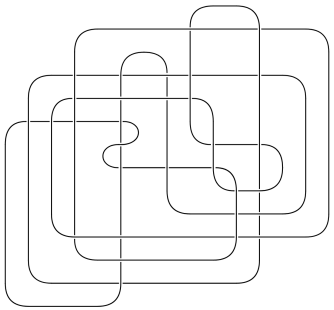

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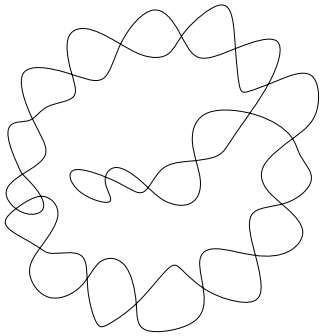

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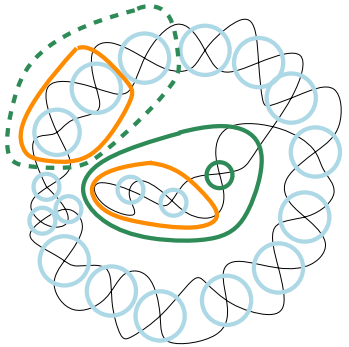

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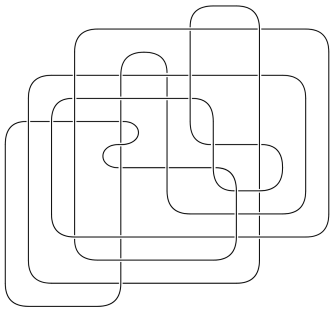

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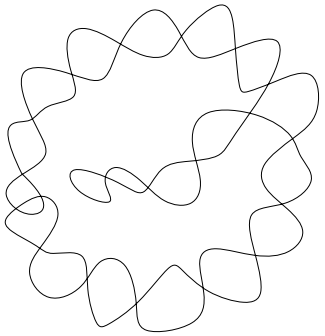

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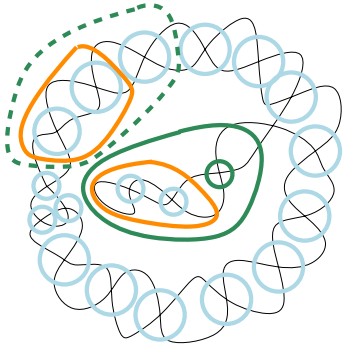

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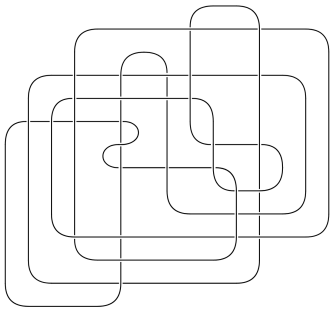

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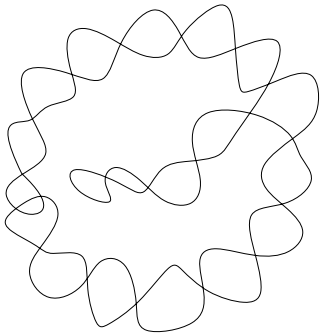

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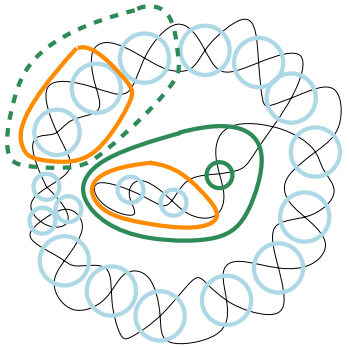

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Many 3-manifolds/knots have cste width: SFS, 2-bridge knots, etc.

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Theorem (de Mesmay, Purcell, Schleimer, Sedgwick '19)

There are families of knots, such as torus knots, with unbounded width.

Overall strategy

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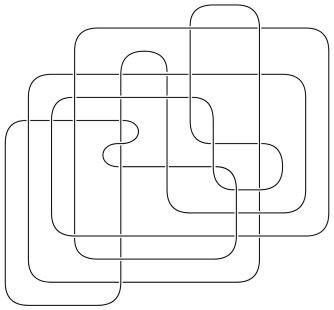

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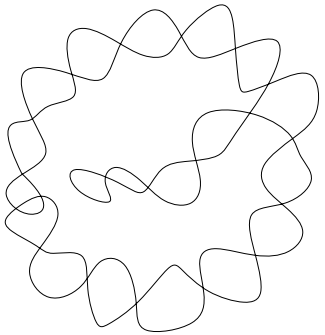

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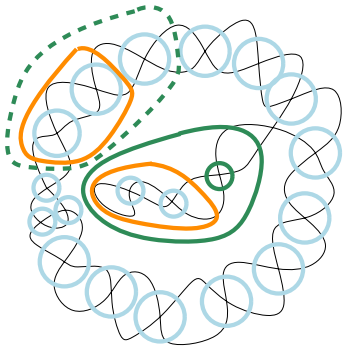

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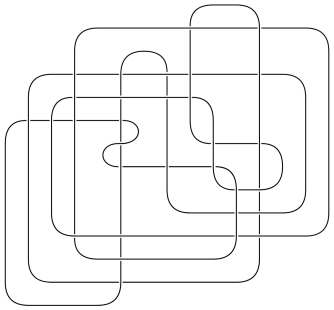

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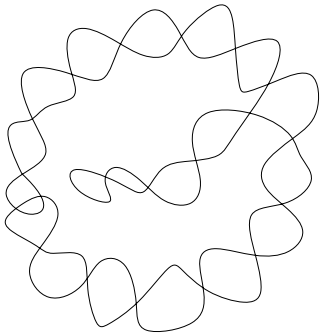

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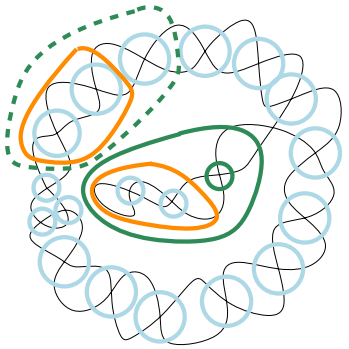

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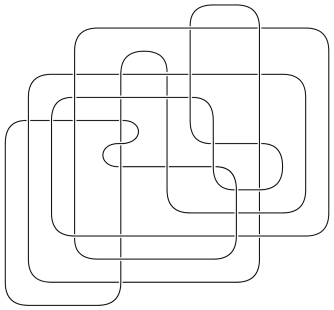

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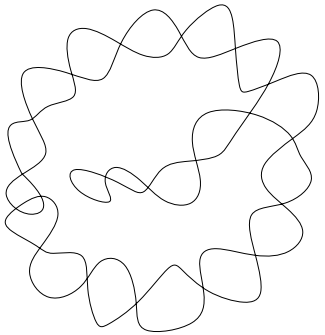

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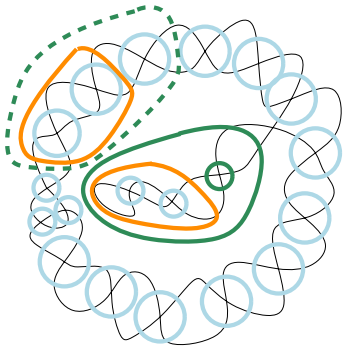

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$\xrightarrow{\text{poly/NP/sub-exp. ?}}$

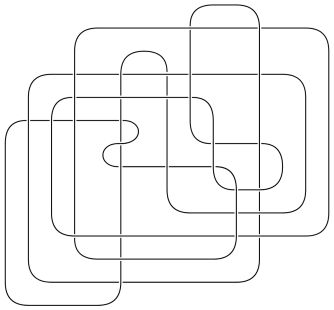
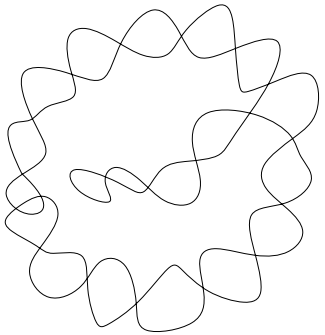
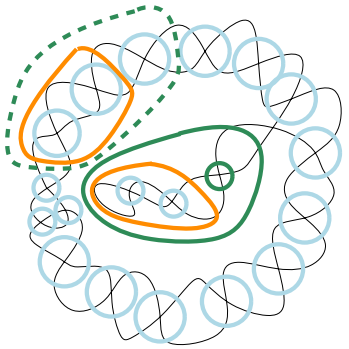


Diagram of K,
- $\mathrm{poly}(n)$ crossings,
- width $f(\mathrm{cw}(K))$ "close to opt".

$\xrightarrow{\text{poly/FPT in width}}$



Computation of a (close to optimal) width-decompostion $g \circ f(\mathrm{cw}(K))$.



FPT algorithm in the width $O(h(\mathrm{cw}(k)) \times \mathrm{poly}(n))$ for a "hard" computation (e.g., $J_N(K)$).

Combine with a bound of the type "$\mathrm{cw}(M) \leq c \times \mathrm{vol}(M)$" for hyperbolic manifolds to get polynomial time algorithms for all hyperbolic manifolds of bounded volume for otherwise hard problems.

Thank you!

Overall strategy

Any diagram of a knot K,
- n crossings,
- arbitr. width.

$\xrightarrow{\text{poly/NP/sub-exp. ?}}$

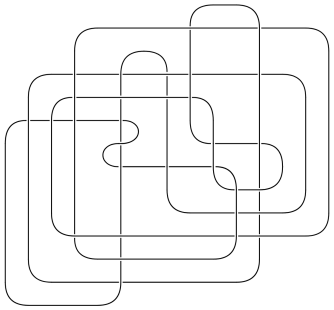
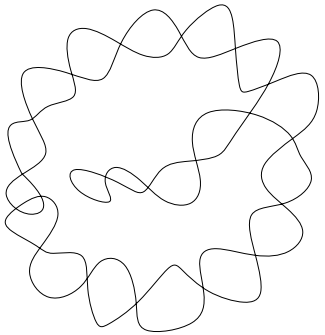


Diagram of K,
- $\mathrm{poly}(n)$ crossings,
- width $f(\mathrm{cw}(K))$ "close to opt".

$\xrightarrow{\text{poly/FPT in width}}$



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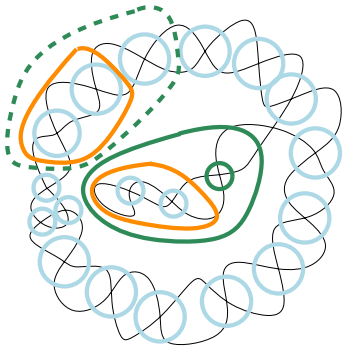


FPT algorithm in the width $O(h(\mathrm{cw}(k)) \times \mathrm{poly}(n))$ for a "hard" computation (e.g., $J_N(K)$).

Combine with a bound of the type "$\mathrm{cw}(M) \leq c \times \mathrm{vol}(M)$" for hyperbolic manifolds to get polynomial time algorithms for all hyperbolic manifolds of bounded volume for otherwise hard problems.

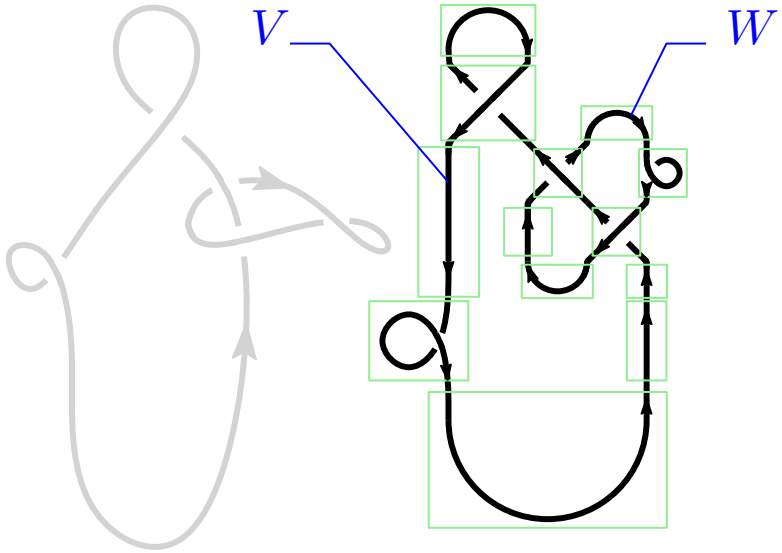
Thank you!

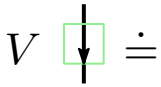
Appendix:
Dynamic programming

Penrose functor: Diagram → (algebraic) invariant

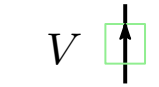
[Sketch of construction]

[Reshetikhin,Turaev]

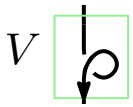




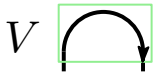
$\doteq \mathrm{id}_V : V \to V$



$\doteq \mathrm{id}_{V^*}$

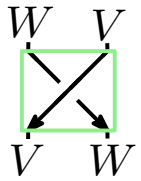


$\theta_V : V \to V$

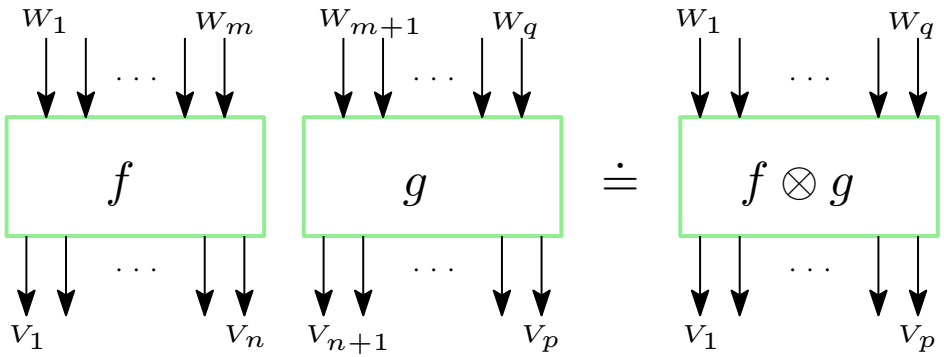


$d_v : V^* \otimes V \to \mathbb{1}$

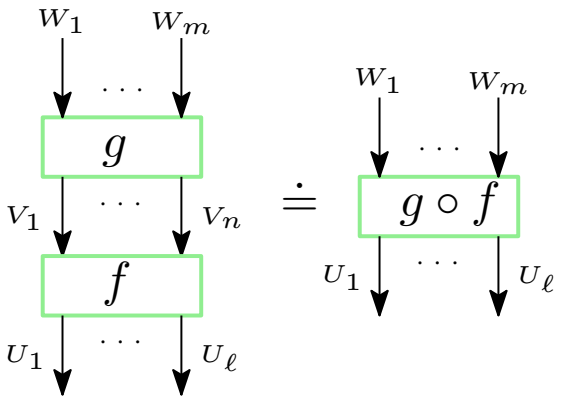
$b_v : \mathbb{1} \to V \otimes V^*$



$c_{V,W} : V \otimes W \to W \otimes V$

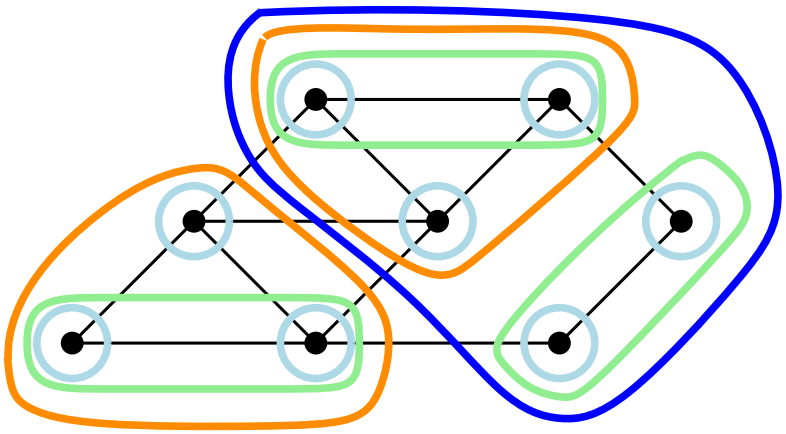


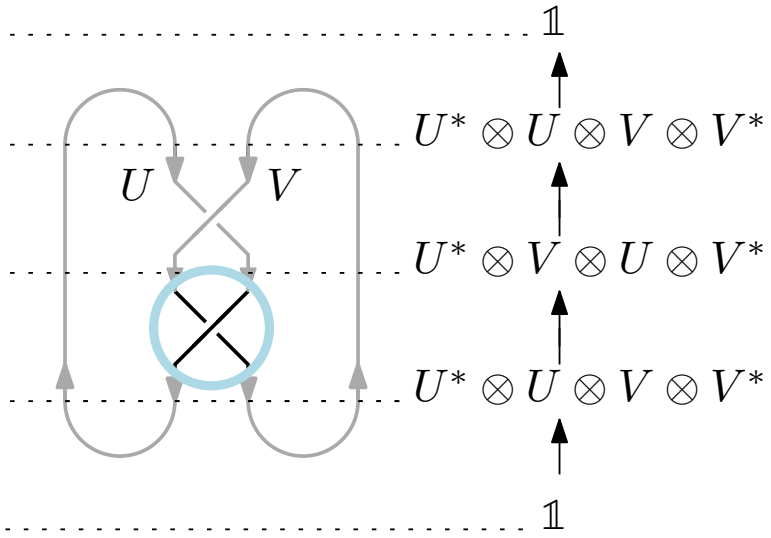
$f \otimes g : V_1 \otimes \ldots \otimes V_p \to W_1 \otimes \ldots W_q$



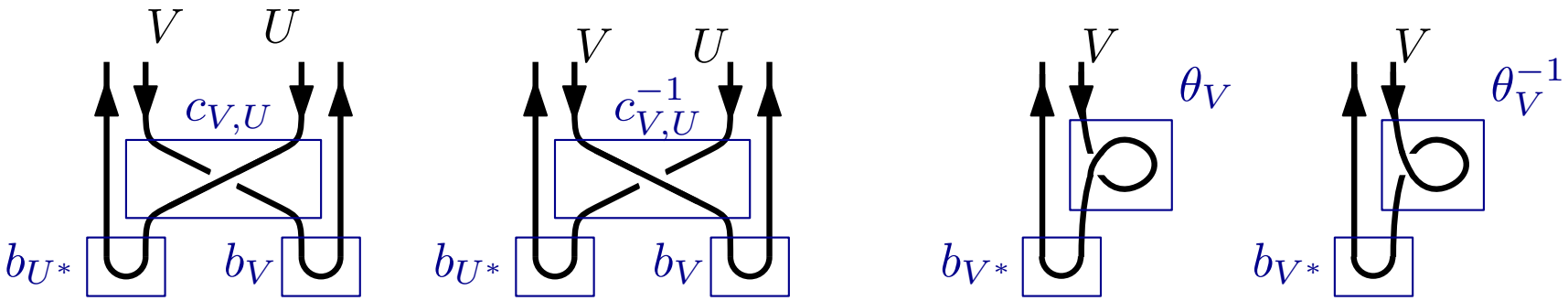
$g \circ f : U_1 \otimes \ldots \otimes U_\ell$

Leaves of the carving decomposition





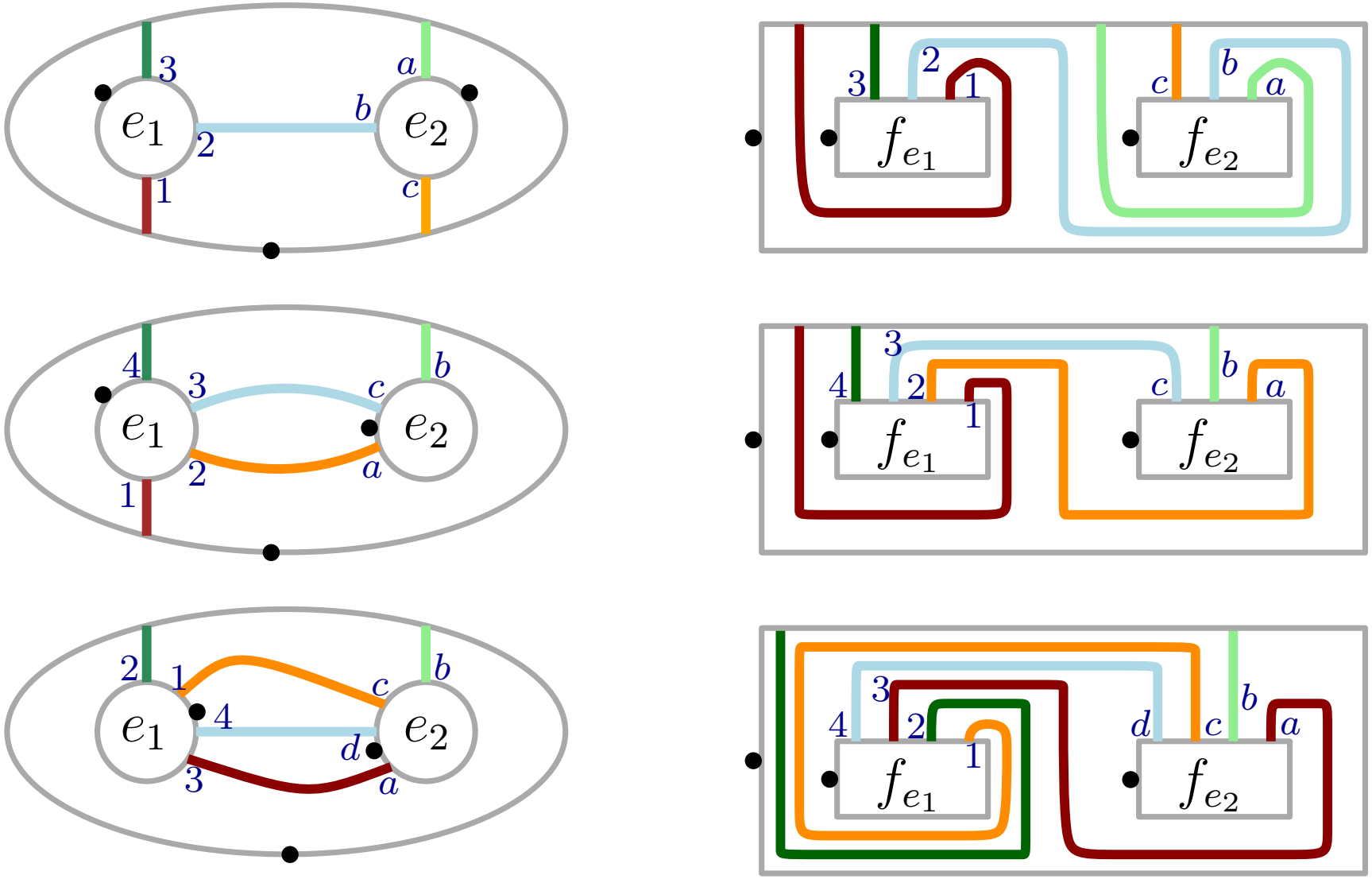
Isotope the link to get morphisms $\mathbb{1} \to W$:



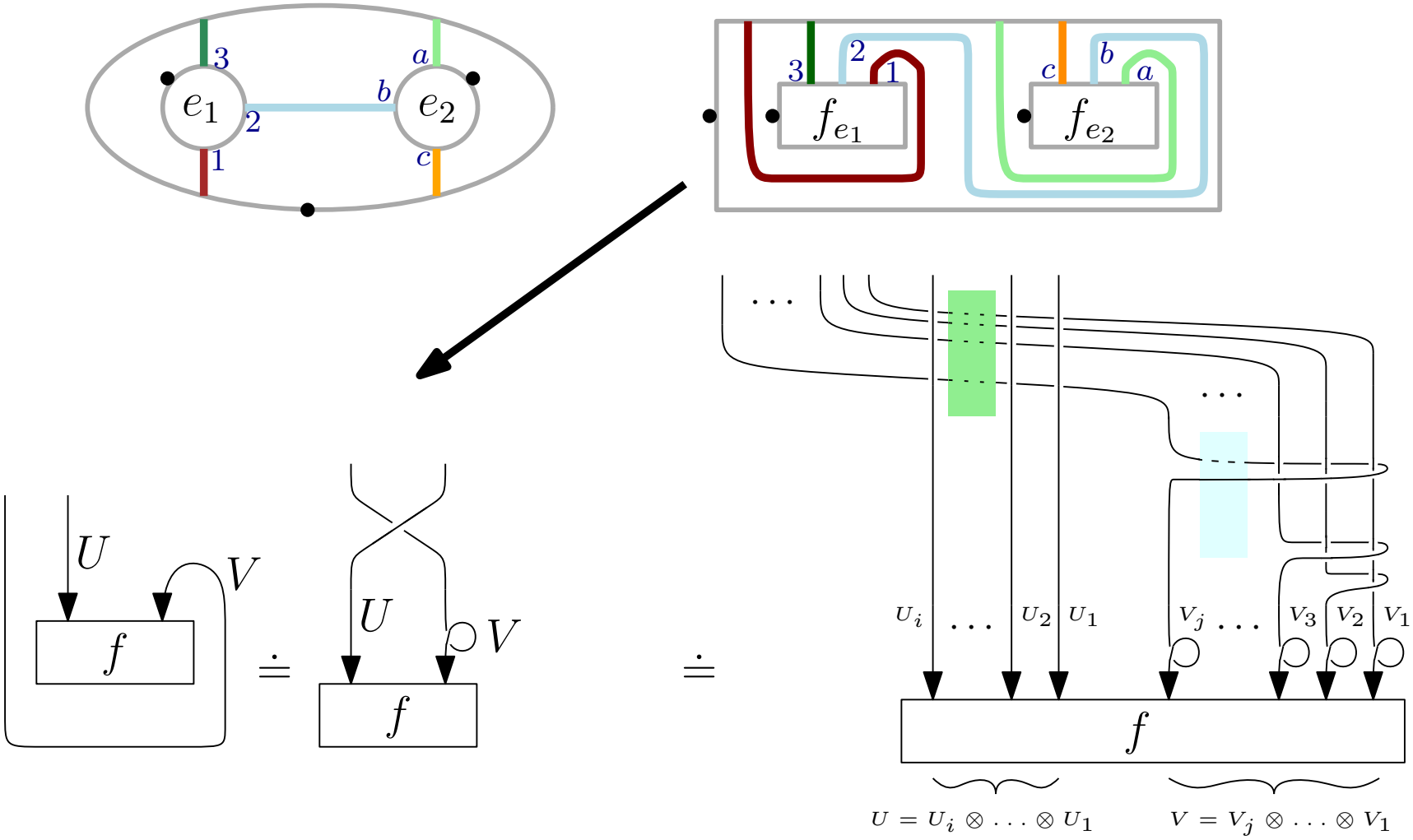
- Four possibilities: two crossings, two twists.
- Constant number of matrix multiplications.

Merging morphisms at tree nodes I

Isotope the link to get morphisms $\mathbb{1} \to W$:

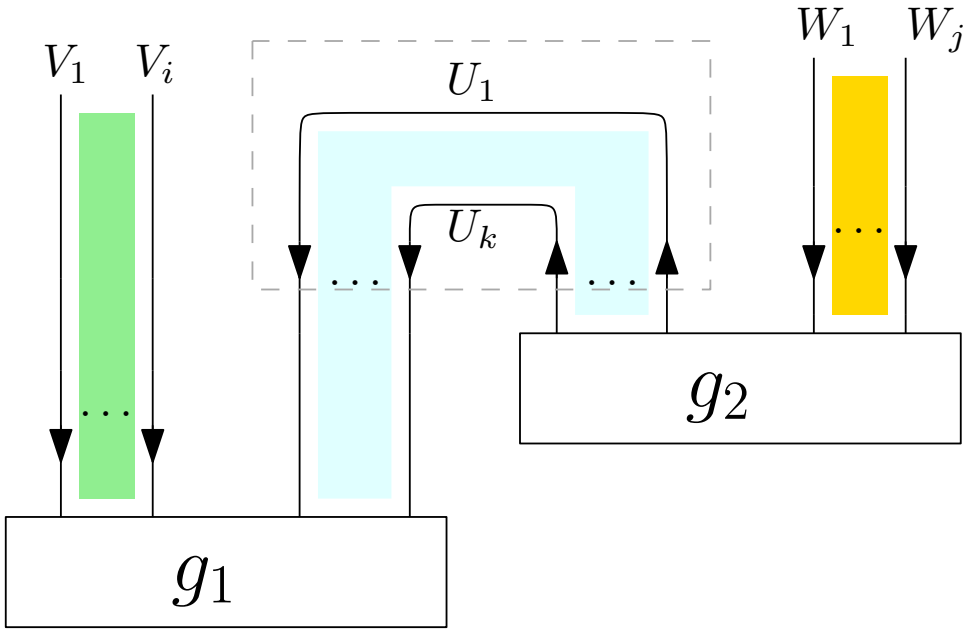


Merging morphisms at tree nodes II



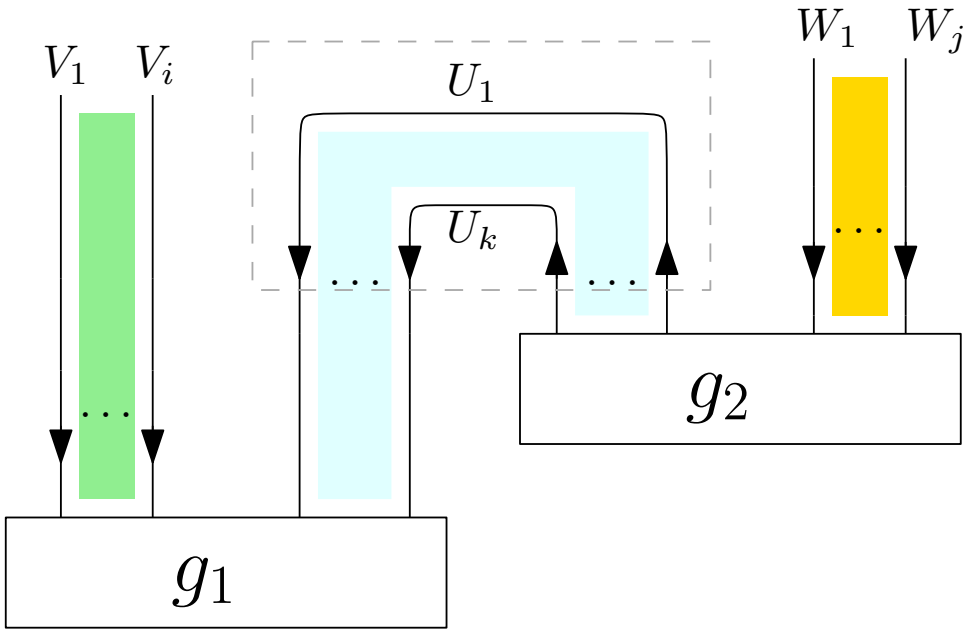
Slide strands under by isotopy $\longrightarrow$ factorise with $O(\mathrm{cw}^2)$ additional matrices.

Merging morphisms at tree nodes III



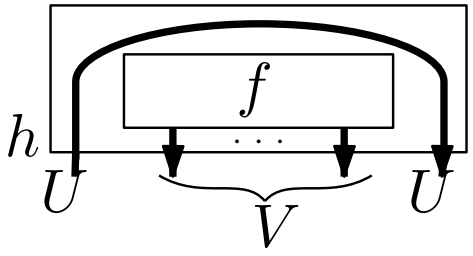
- $i + j \leq \mathrm{cw}$,
- can assume $k \leq \mathrm{cw}/2$.

Merging morphisms at tree nodes III



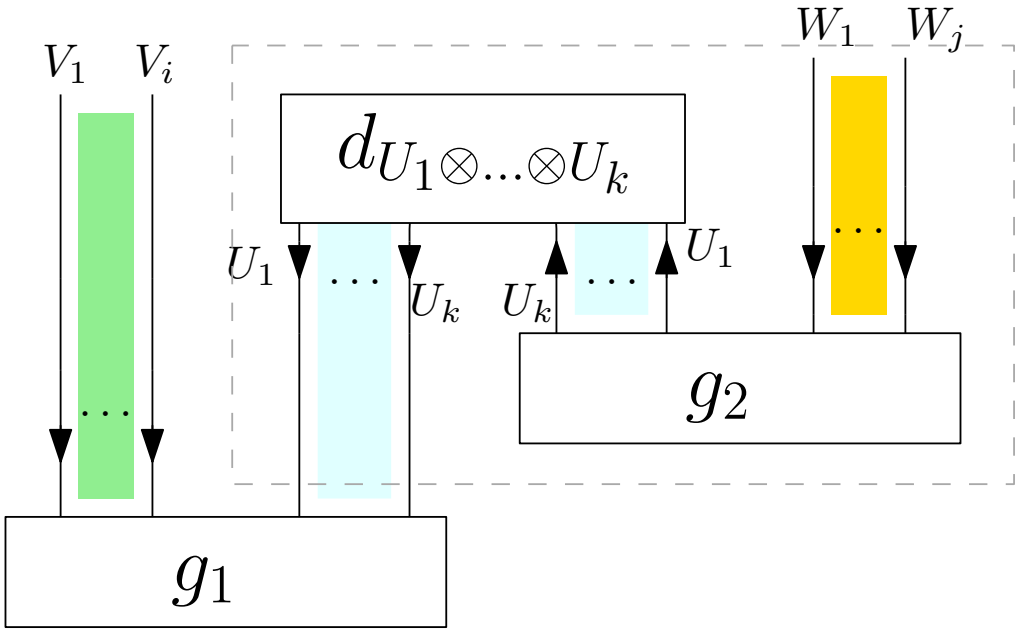
- $i + j \leq \mathrm{cw}$,
- can assume $k \leq \mathrm{cw}/2$.

Operation (k times):

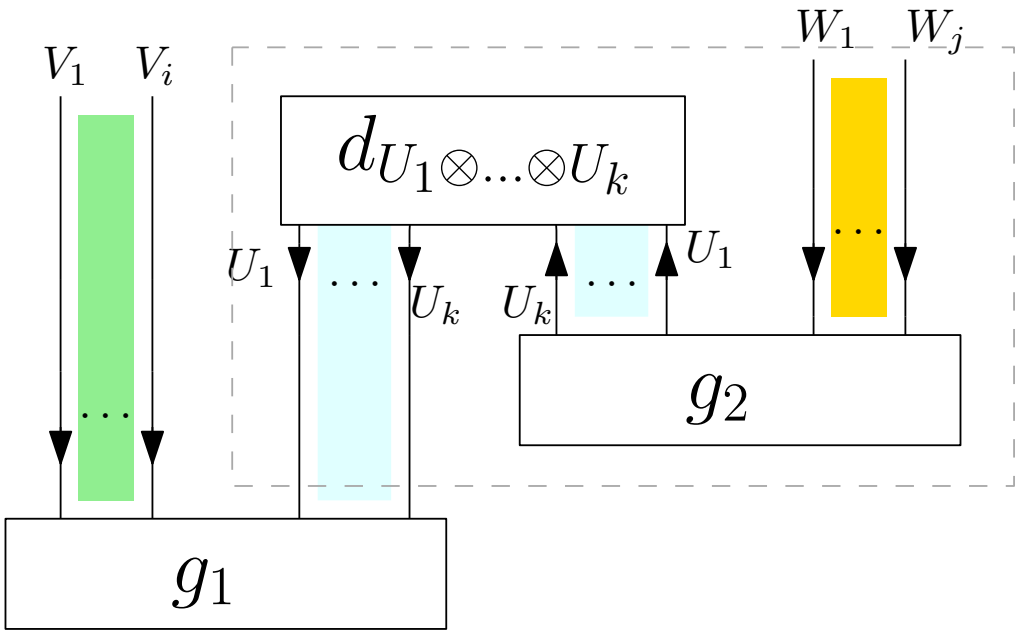


Next $\rightarrow$ factorise the "U_i-bridge".

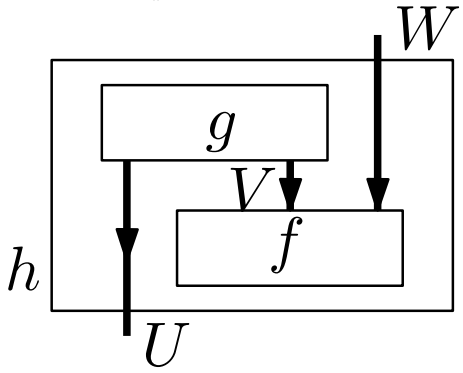
Merging morphisms at tree nodes III



Merging morphisms at tree nodes III

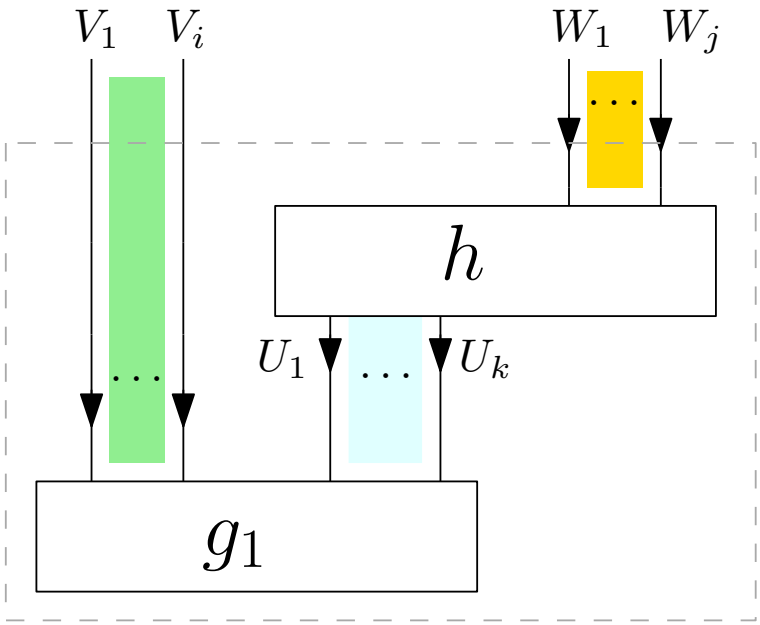


Operation:

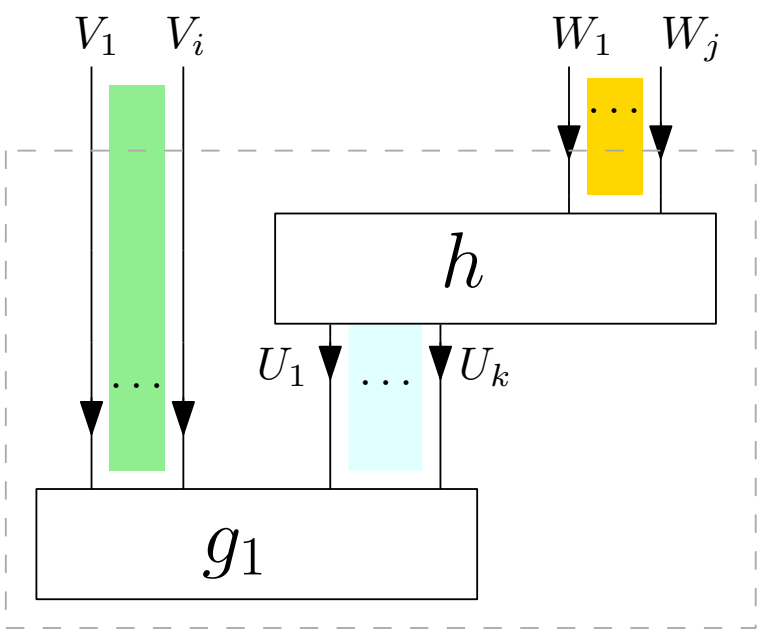


Next $\rightarrow$ factorise $d_{U_1 \otimes \ldots \otimes U_k}$ and g_2.

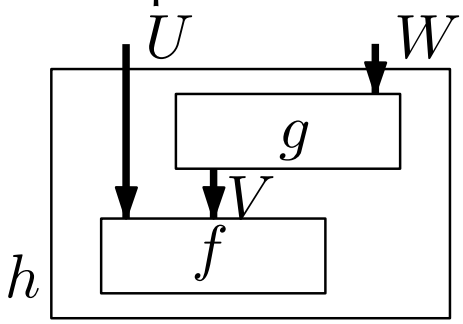
Merging morphisms at tree nodes III



Merging morphisms at tree nodes III

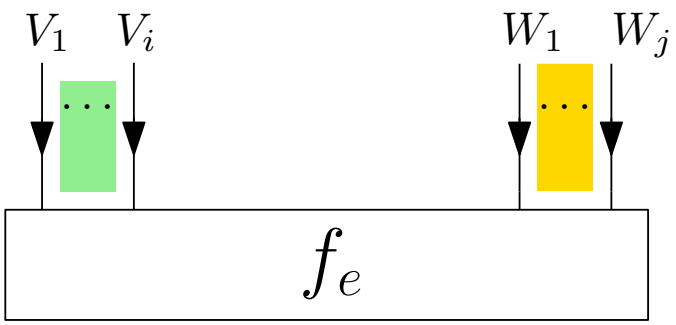


Operation:

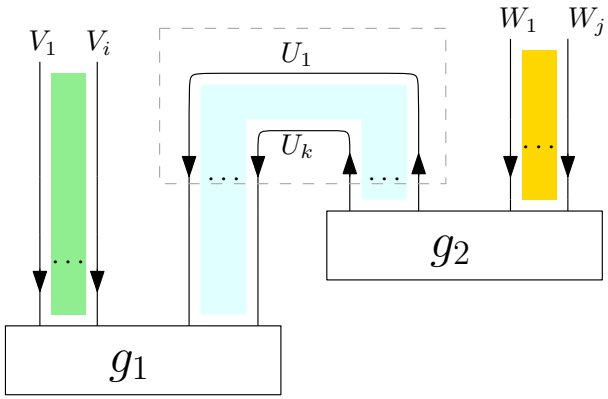


Next $\rightarrow$ factorise h and g_1.

Merging morphisms at tree nodes III



Complexity



- $i + j \leq \mathrm{cw}$,
- can assume $k \leq \mathrm{cw}/2$.

- All op. are sparse matrix multiplications, of type: $\mathrm{id} \otimes M \otimes \mathrm{id}$.
- Matrices are all of size $N^{O(\mathrm{cw})} \times N^{O(\mathrm{cw})}$, as morphisms of type

$$U_1 \otimes \ldots \otimes U_{O(\mathrm{cw})} \to V_1 \otimes \ldots \otimes V_{O(\mathrm{cw})}, \textit{ with } \dim U_i, \dim V_j \leq N.$$

- Can control the arithmetic complexity of operations in the ring R (e.g., $R = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$).

Complexity

Theorem (M. '21)

Fix a strict ribbon category $\mathcal{C}$ of $\mathbb{Z}[X, X^{-1}]$-modules, and free modules $V_1, \ldots, V_m \in \mathcal{C}$ of dimension bounded by N. The problem:

> Quantum invariant at $\mathcal{C}, V_1, \ldots, V_m$:
> **Input**: *m-components link L, presented by a diagram $D(L)$,*
> **Output**: *quantum invariant $J_L^{\mathcal{C}}(V_1, \ldots, V_m)$*

can be solved in

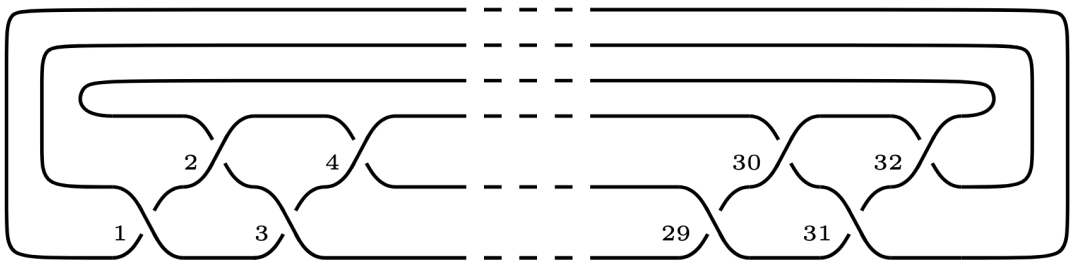
- $O(\mathrm{poly}(n) \cdot N^{\frac{3}{2}\,\mathrm{cw}})$ *machine operations, with*
- $O(N^{\mathrm{cw}} + n)$ *memory words,*

where n and cw are respectively the number of crossings and the carving-width of the diagram $D(L)$.

NB: $\mathrm{cw} = O(\sqrt{n}) \Rightarrow$ sub-exponential algo.

Experiments

- Beautiful and difficult knot (weaving knot $W(3,16)$):



- Exciting conjecture (volume conjecture):

$$\underbrace{\lim_{\mathbf{N}\to+\infty} \frac{2\pi \log\left| J_{\mathbf{N}}(K)(e^{2\pi i/\mathbf{N}}) \right|}{\mathbf{N}}}_{\text{limit behaviour of coloured Jones polynomials}} = \underbrace{\mathrm{vol}(S^3 \setminus K)}_{\text{hyperbolic volume of the knot}}$$

- Very early experiments (some convergence of the sequence above):