Tree-like width of knots and obstructions.

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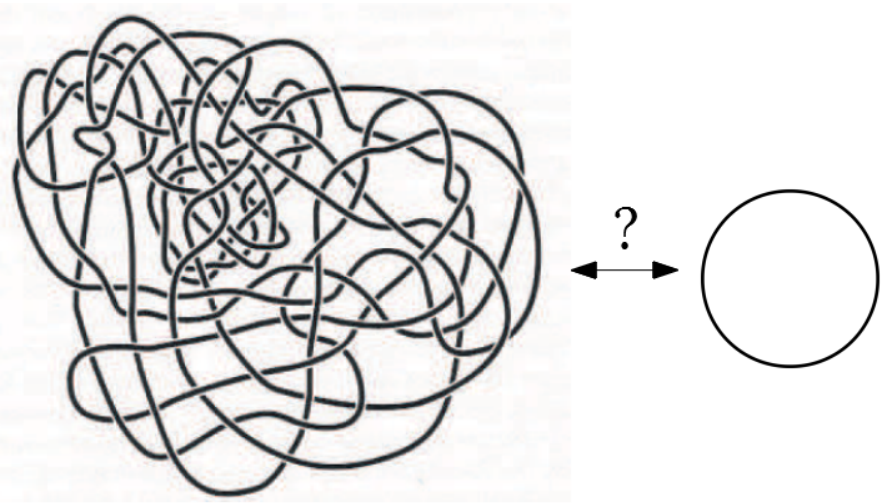
Joint work with Arnaud de Mesmay.
Université Gustave Eiffel, LIGM

Knots

Knot :

A knot is a tame embedding $\mathbb{S}^1 \to \mathbb{S}^3$ considered up to continuous deformation (ambient isotopy).



Natural algorithmic questions :

- Is a given knot equivalent to the trivial one ? (**NP** and **co-NP**)

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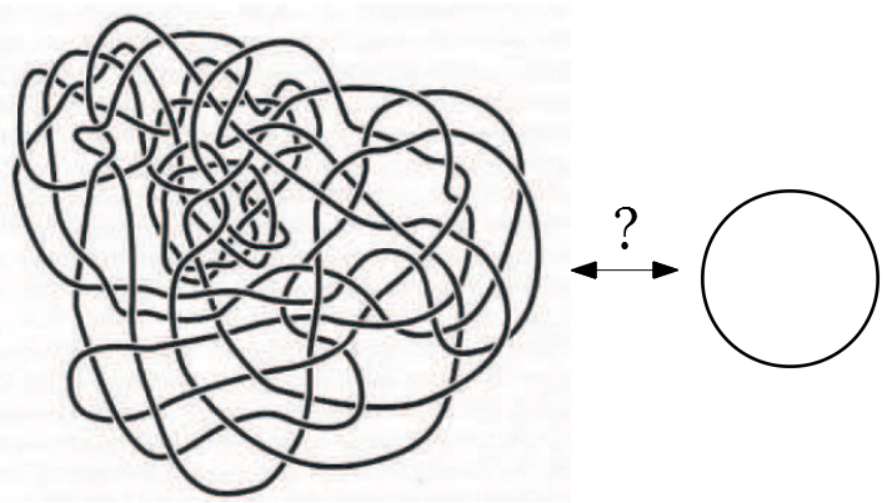


Natural algorithmic questions :

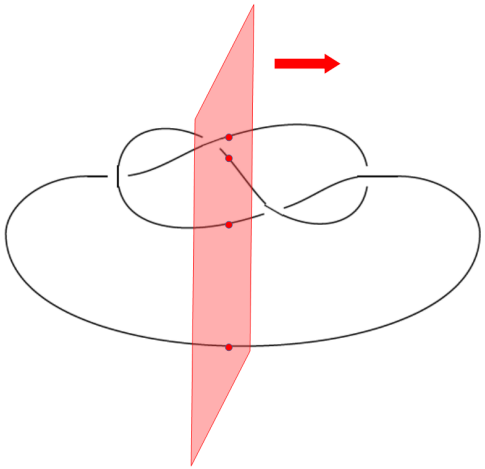
- Is a given knot equivalent to the trivial one ? (**NP** and **co-NP**)
- Are two given knots equivalent ?

"Width" invariant of knots

Aim to measure some notion of "width" on knots.

Examples of width invariants :

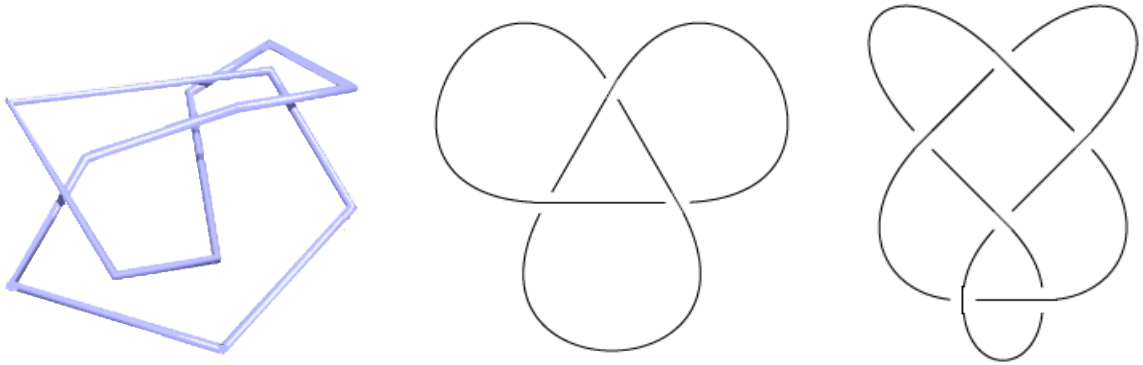
- Trunk : Minimal number among linear sweepouts of the maximal number of intersections between the knot and a plane (or a sphere).



- Bridge number : Minimal number of trivial arcs on a sphere that "splits" the knot.

Treewidth on knots

Aim to measure "how close" a knot is to a tree.



A diagram of a knot is a 4-valent graph.

Diagrammatic treewidth of a knot K :

It is the minimum treewidth of a diagram of K.

Treewidth of a graph

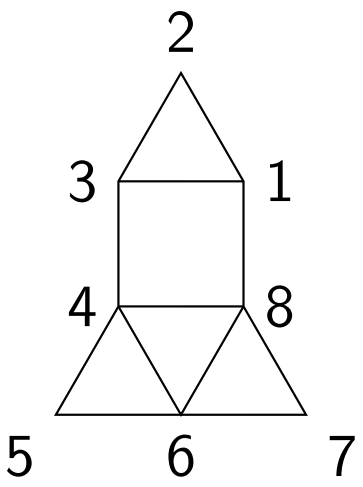
Aim to measure "how close" a graph is to a tree.

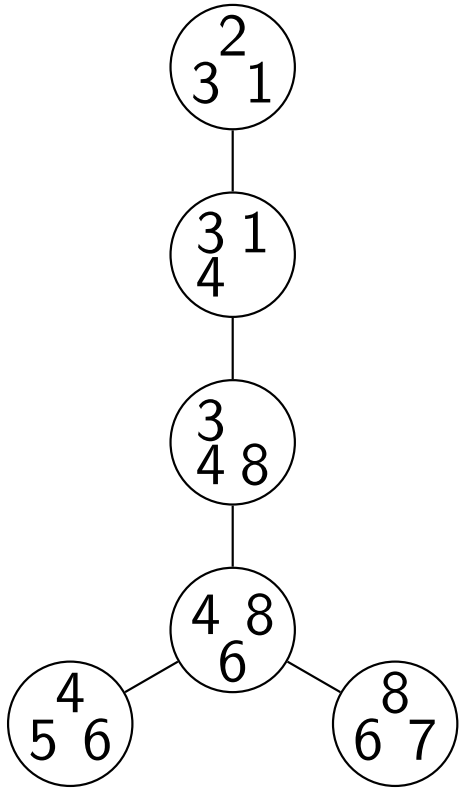
Treewidth of a graph G :

Maximum size of a bag in a tree decomposition of G.

In a tree decomposition :

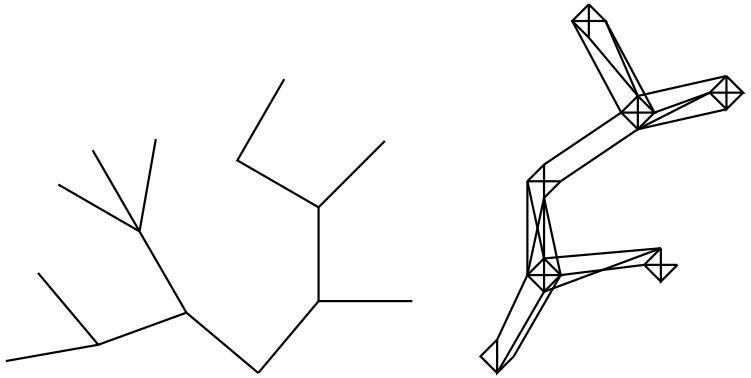
- Each vertex is in a bag
- Each pair of edge extremities is in a bag
- Bags containing a fixed vertex create a subtree



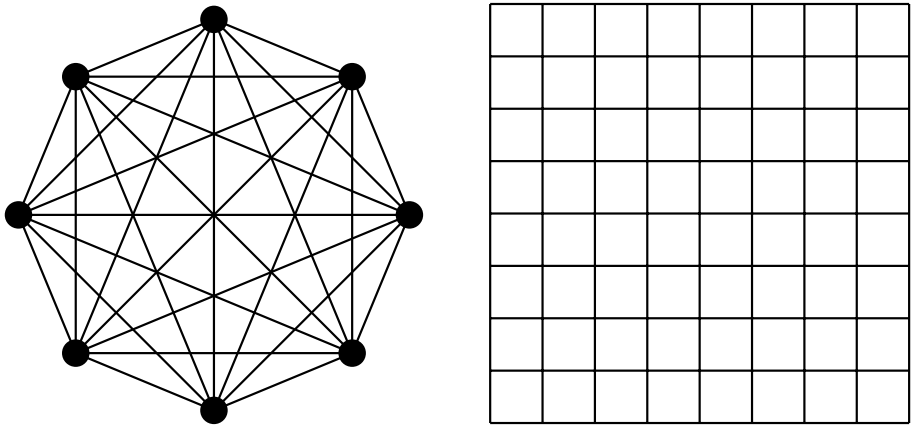


Treewidth of a graph

Small treewidth :



High treewidth :



Why is treewidth interesting?

- Key tool for **dynamic programming** on graphs.
  For example: Jones polynomial, Kauffman polynomial [J. A. Makowsky and J. P. Mariño, 2003], HOMFLY-PT polynomial [B. A. Burton, 2018], and quantum invariants [C. Maria, 2019] are efficiently computable on small treewidth diagrams of knots.

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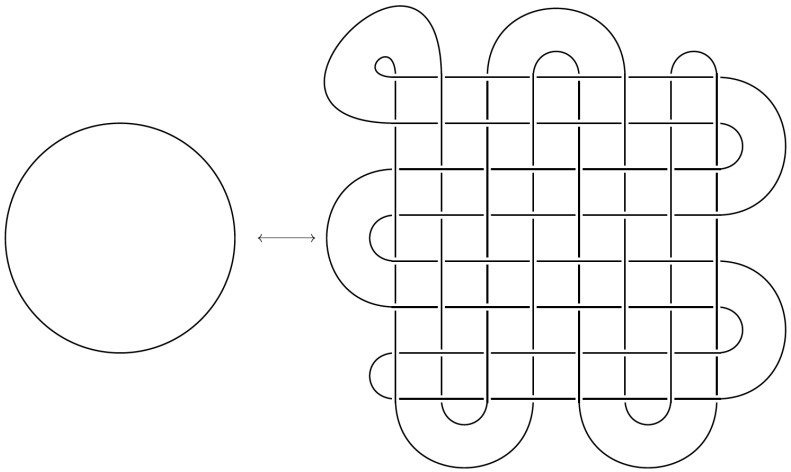
- It plays a major role in the **graph minor theory** from Robertson and Seymour.
- Occurs naturally in other areas:
  For example: 3-manifolds [K. Huszár and J. Spreer, 2019], lower bounds distortion on knots...

Back to knots

Knots always have diagrams with high treewidth.



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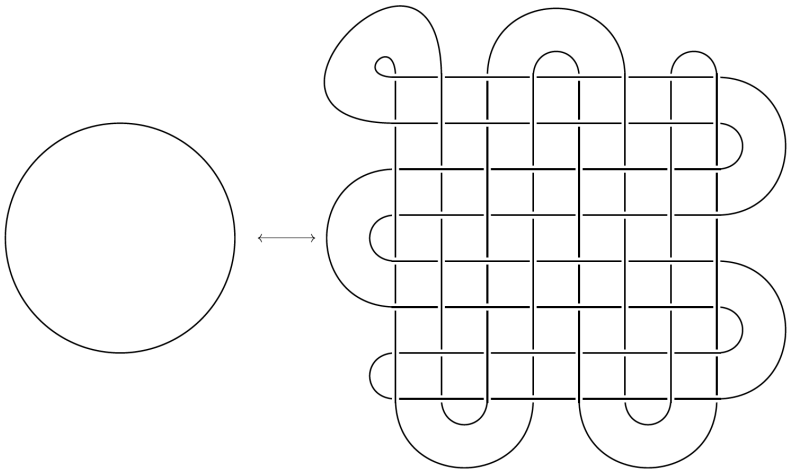


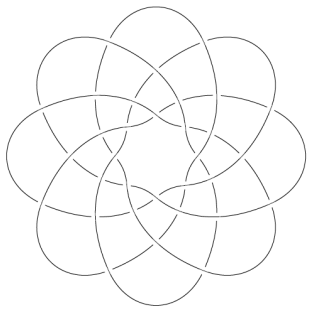
Question from [J. A. Makowsky and J. P. Mariño., *The parametrized complexity of knot polynomials*, 2003] :

Are there knots for which all diagrams have high treewidth ?

Back to knots

Theorem [A. De Mesmay, J. Purcell, S. Schleimer, and E. Sedgwick, *On the tree-width of knot diagrams.*, 2019]:

Let $T_{p,q}$ be a torus knot. Then $\mathrm{tw}(T_{p,q}) = \Omega(\min(p,q))$.



Towards a 3D definition:

The proof uses a width notion generalising carving width (treewidth equivalent) on knots, using multiple Heegaard splittings.

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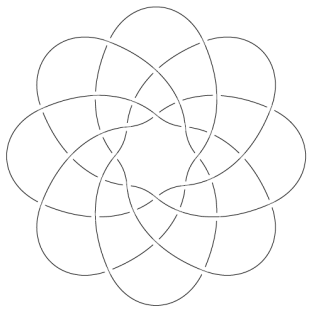

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My goal:

Define a width invariant on knots generalising branchwidth: definition stemming from structural graph theory tailored to proving lower bounds.

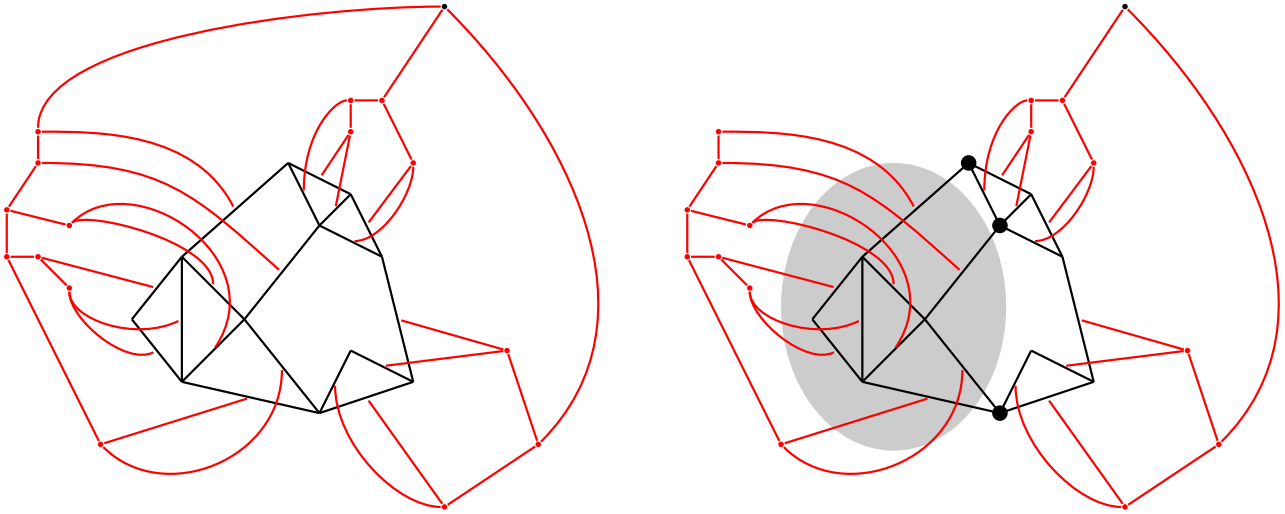
Branchwidth definition

Branchwidth is another equivalent of treewidth : $\frac{2}{3}tw \leq bw \leq tw$.

Branchwidth of a Graph :

T is a trivalent tree and $\phi : E(G) \to L(T)$ is a bijection.

$$\mathrm{bw}(G) = \min_{(T,\phi)\in \mathrm{BD}(G)} \max_{e\in T} |\partial\phi^{-1}(L(T_1))|$$



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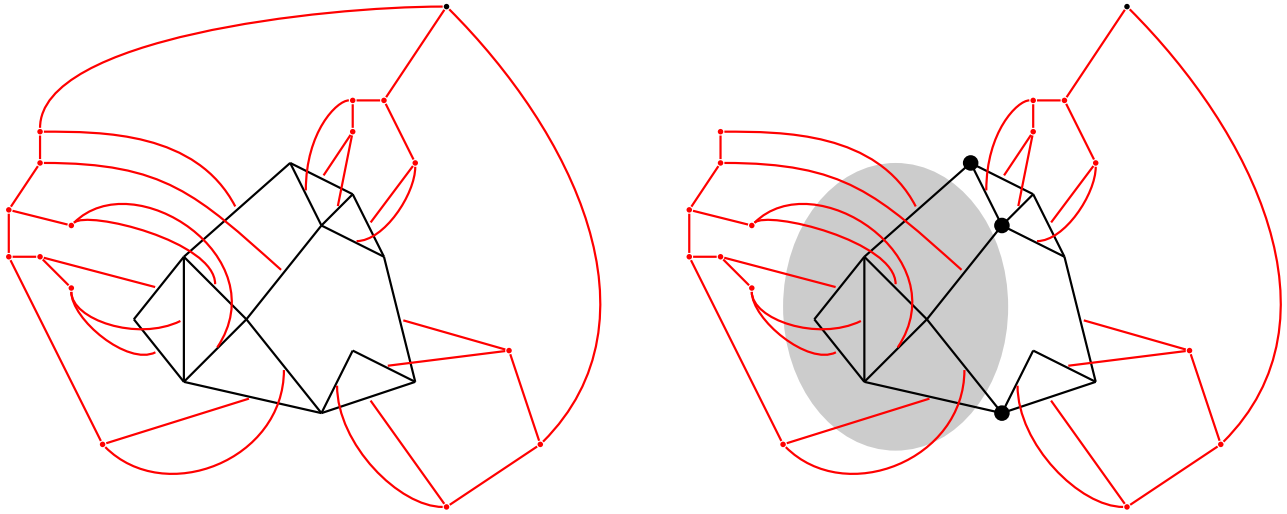


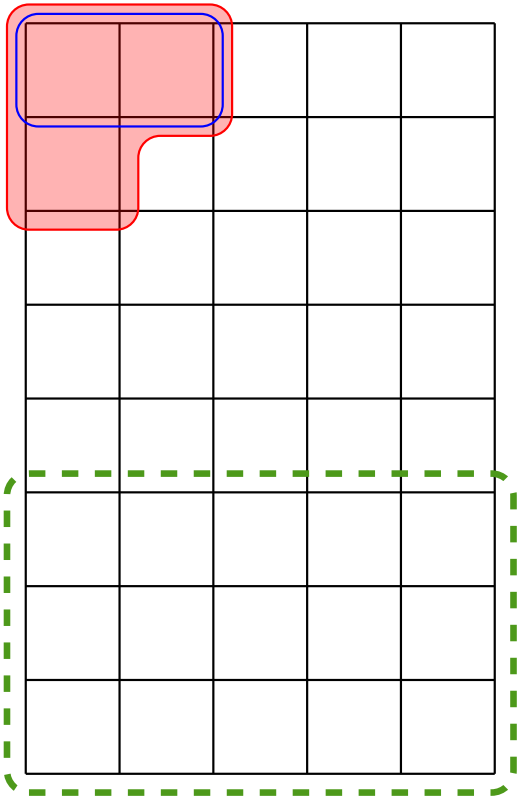
Branchwidth can be **lower bounded** through its dual problem: existence of a tangle.

Tangle on graphs

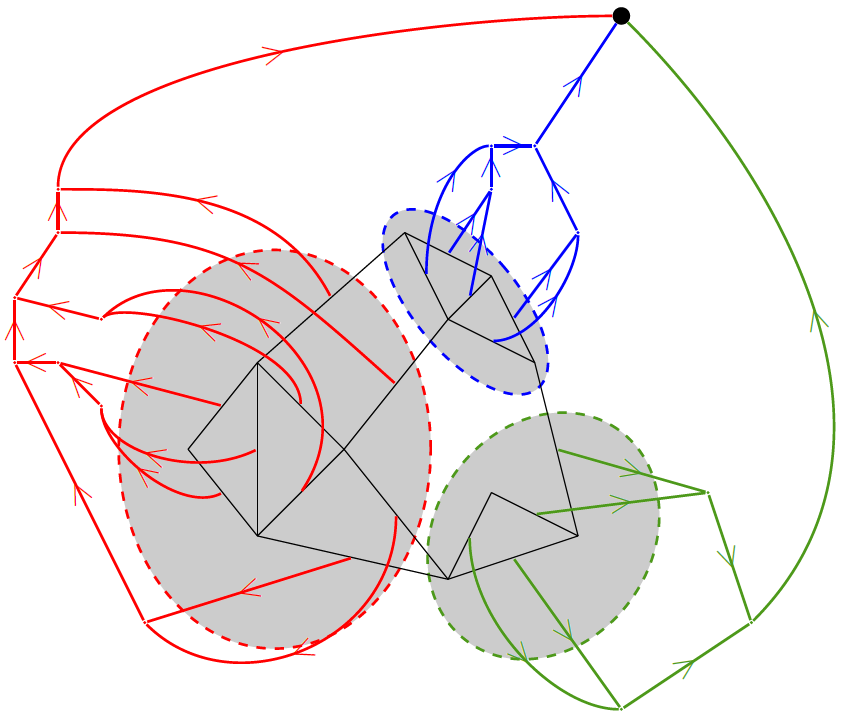
Tangle definition

A **tangle of order** k, noted $\mathcal{T} \subset \mathcal{P}(E)$is a collection of subsets of E ("small sides") such that :

- $\forall A \in \mathcal{T}, \partial A < k$, sets in the tangle have boundary less than k.
- For all $A, B \in \mathcal{P}(E)^2$, if (A, B) is a separation of order less than k, $\mathcal{T}$ contains A or B.
- $\forall A \in \mathcal{P}(E), \forall B \in \mathcal{T}, A \subset B$ and $\partial A < k \Rightarrow A \in \mathcal{T}$. Small sides are consistent.
- $\forall A, B, C \in \mathcal{P}(E)^3, A \sqcup B \sqcup C = E$ $\Rightarrow \{A, B, C\} \not\subset \mathcal{T}$. No three small sides can cover the whole space.
- $\forall e \in E, E \setminus \{e\} \notin \mathcal{T}$. A single edge is small.

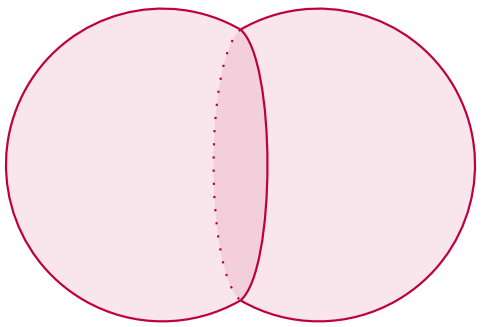


Obstruction



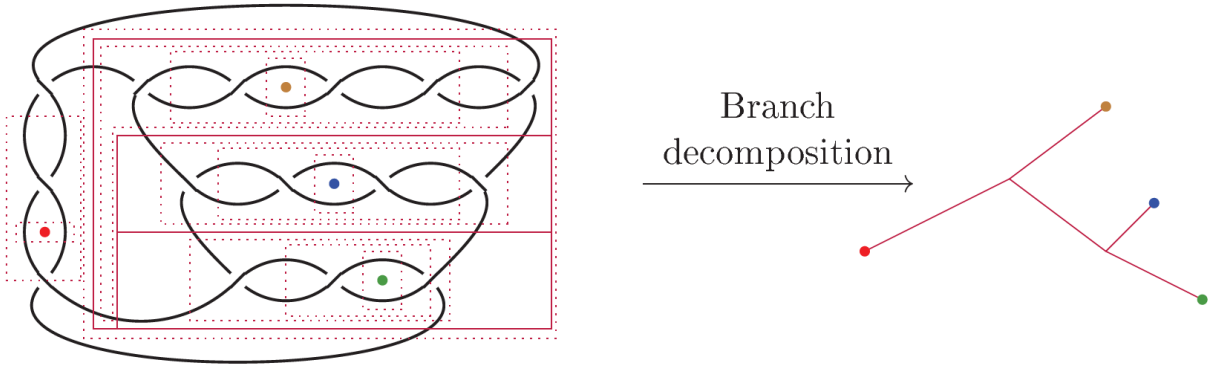
A branch decomposition of width 3 implies the impossibility for a tangle of order 3.

Branchwidth definition



A double bubble : two spheres that intersect on a single disk.

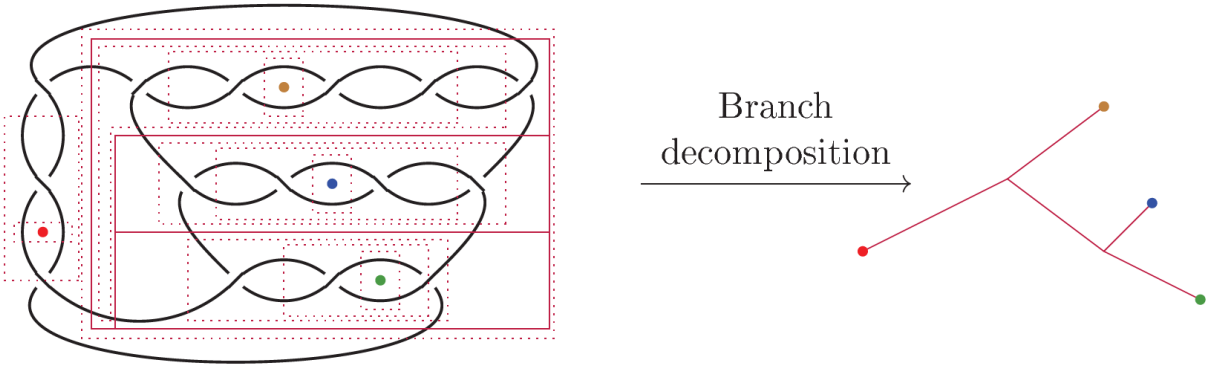
Branchwidth definition



A **branch-decomposition** or **sweep** of $\mathbb{S}^3$ is a continuous map $f : \mathbb{S}^3 \to T$ where T is a trivalent tree such that :

$$f^{-1} : \begin{cases} \text{leaf} & \mapsto \text{point} \\ \text{vertex} & \mapsto \text{double bubble} \\ \text{point interior to an edge} & \mapsto \text{sphere} \end{cases}$$

Branchwidth definition



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Branchwidth

The branchwidth of K written $\mathbf{bw}(K)$ is :

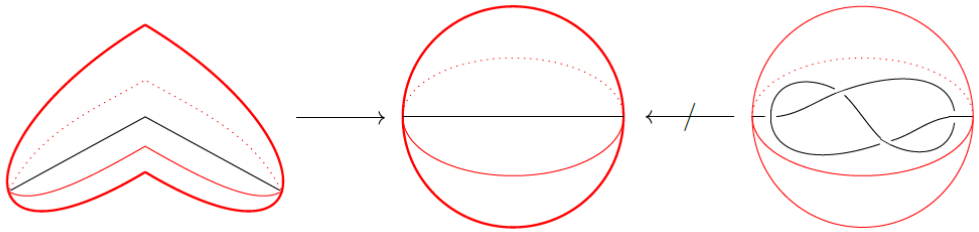
$$\mathrm{bw}(K) = \inf_{f} \sup_{e \in E(T), x \in \mathring{e}} |f^{-1}(x) \cap K|.$$

Bubble tangle on knots

Bubble tangle

Let K be a knot, **tangle of order** n, noted $\mathcal{T}$ is a collection of closed balls of $\mathbb{S}^3$ such that:

- All subsets of $\mathcal{T}$ have less than n intersections with K on their boundaries.
- For any spheres S of $\mathbb{S}^3$, if $|S \cap K| < n$ then a component of $\mathbb{S}^3 \setminus S$ is in $\mathcal{T}$.
- For any three closed balls B_1, B_2, B_3 that induces a double bubble, not all three of B_1, B_2, B_3 are in $\mathcal{T}$.
- T contains trivial balls:



Results

Theorem 1

If there exists a bubble tangle of order n on K then $\mathrm{bw}(K) \geq n$.

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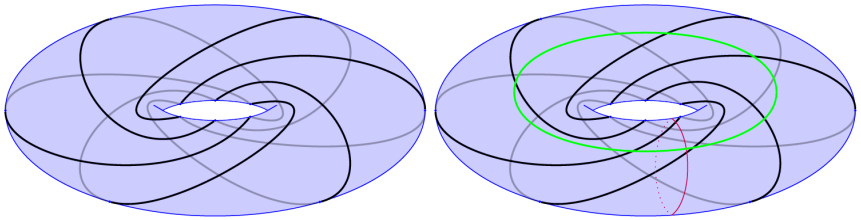
Theorem 2

Let n be the maximal order of a bubble tangle on K. Then $\mathrm{bw}(K) = n$.

Distortion and representativity

Representativity

If $K \hookrightarrow \Sigma \hookrightarrow \mathbb{S}^3$, the representativity of K on Σ is the minimum number of intersection between a non contractible, compressible curve on Σ and K.



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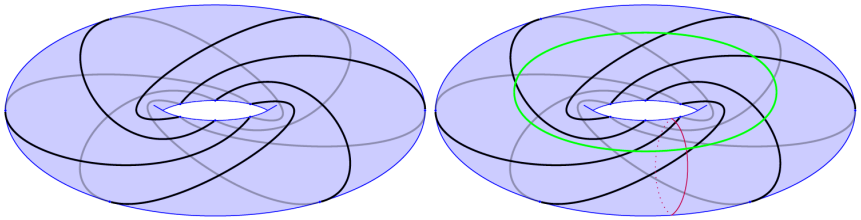


Distortion

The distortion of $K \hookrightarrow \mathbb{R}^3$ is defined as $\delta(K) = \sup_{x,y \in K} \frac{d_K(x,y)}{d_{\mathbb{R}^3}(x,y)}$.

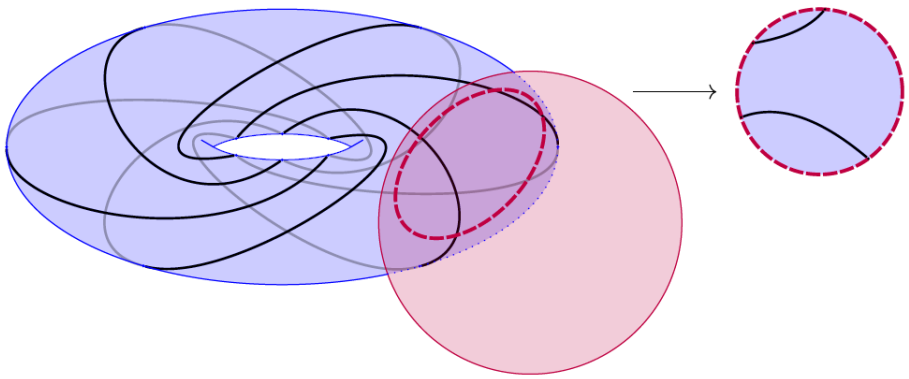
Pardon showed that representativity of torus knots lower bound their distortion [Pardon, 2011].

Bubble tangle from representativity

Generalising the work of [Pardon, 2011],

Theorem 3

Let K be a knot of representativity n on a surface Σ. Then there exist a bubble tangle of order $c \times n$ (where $c \in [\frac{1}{2}, \frac{4}{3}]$).



Corollary

Torus knots have high treewidth.

Conclusion

Results:

- Branchwidth is a width invariant that can be defined on knots and generalises the notion from structural graph theory.
- It can be easily lower bounded through the dual problem: bubble tangle.
- These obstructions can arise from representativity.
- Extends naturally to spatial graphs

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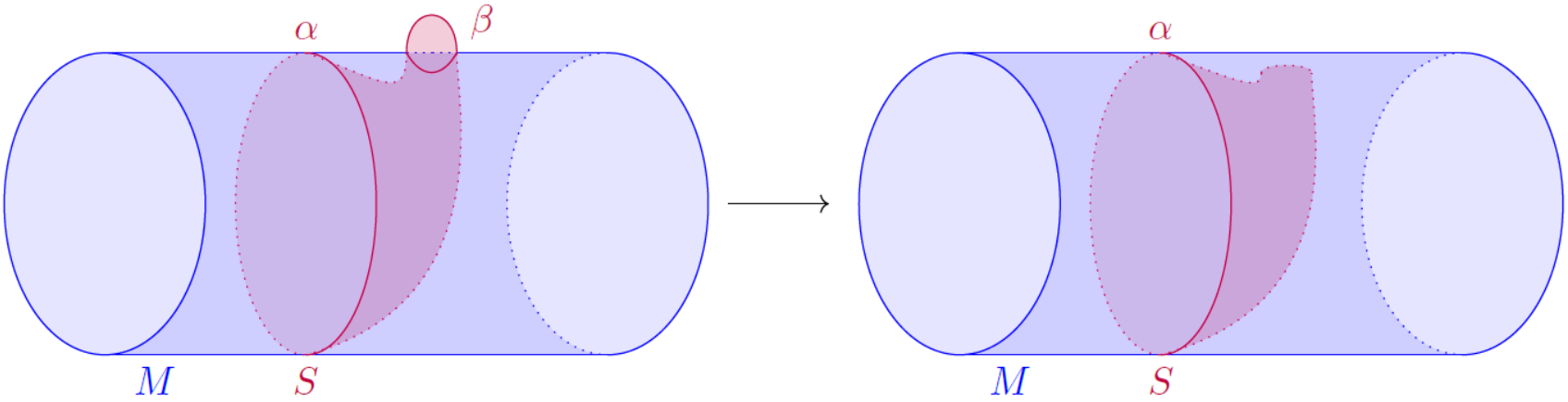
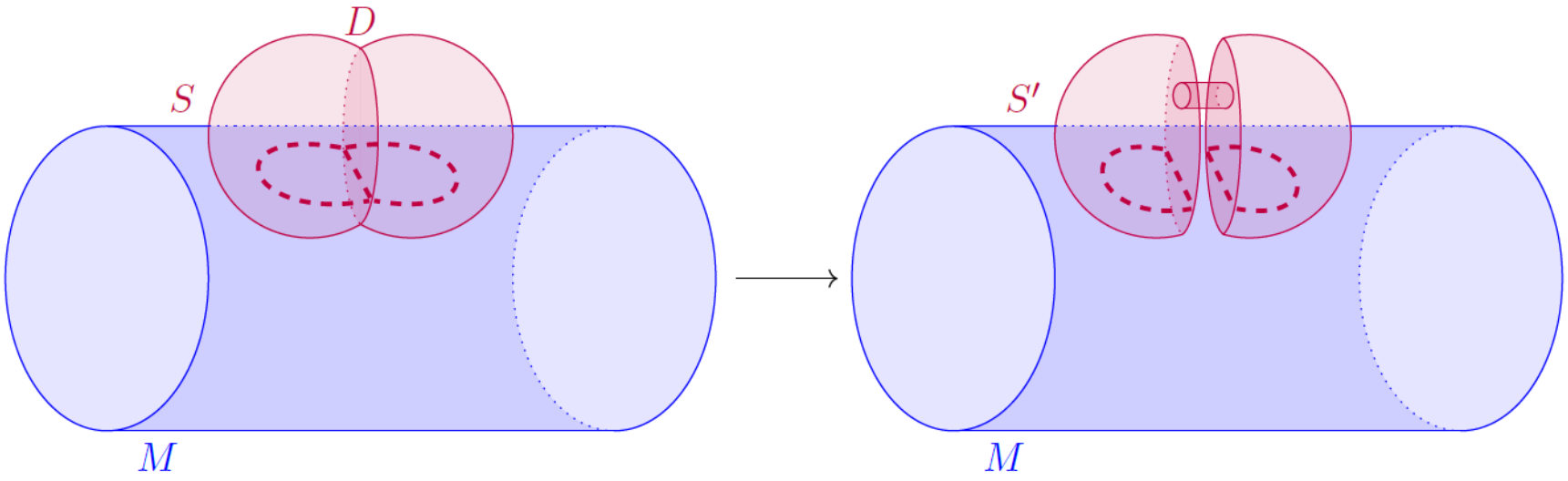
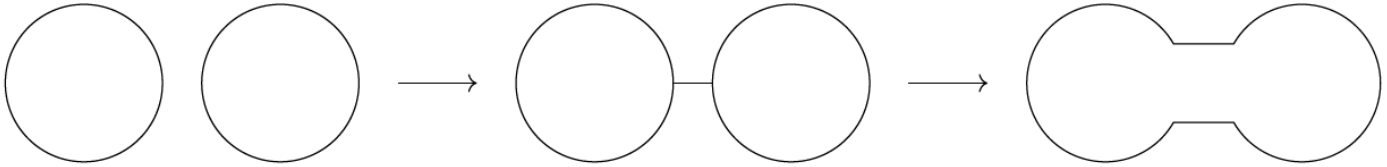
Thank you for listening !



Removing an inessential curve from S.

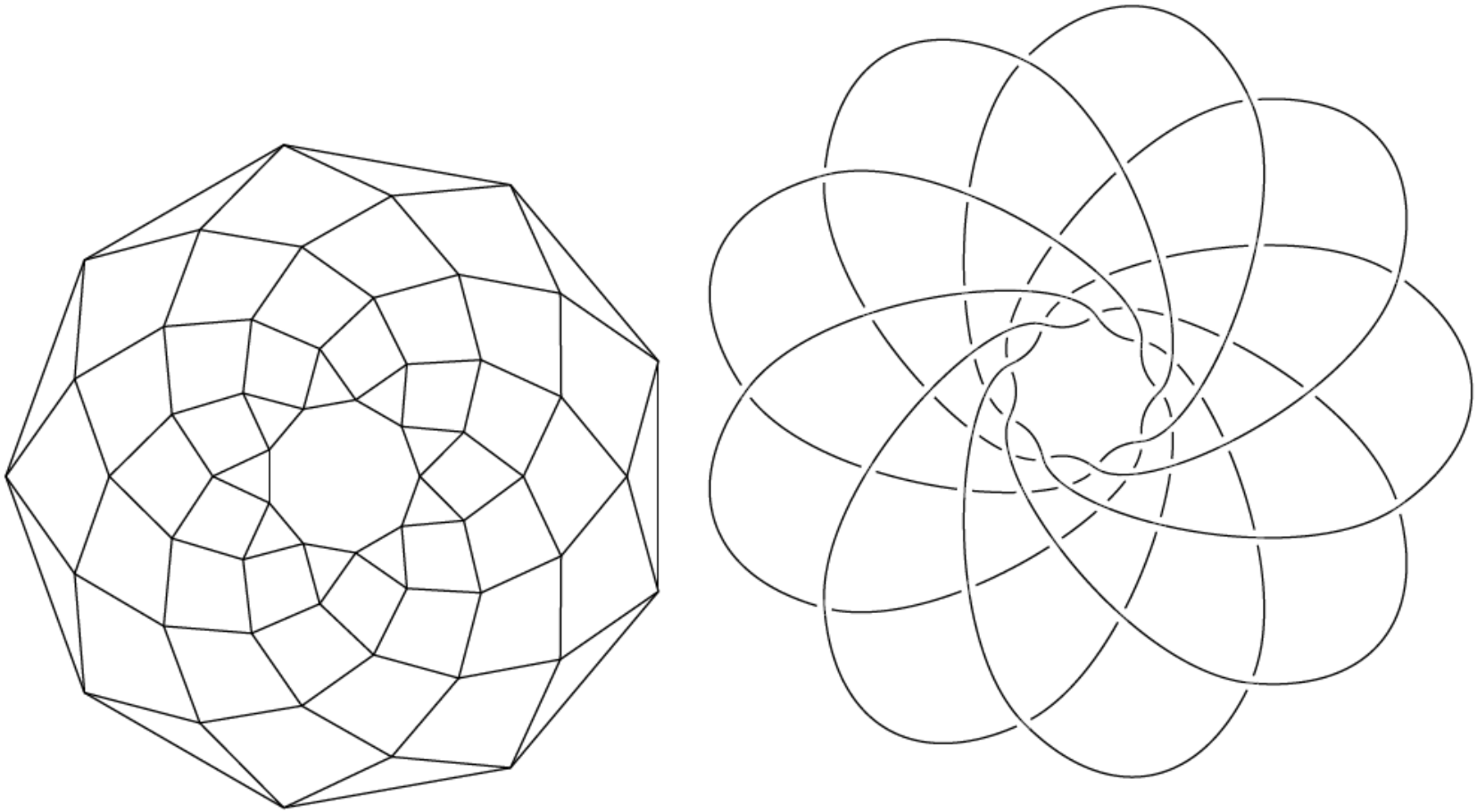


Transformation from a double bubble to S'



Evolution of two circles of $S_t \cap M$ when t grows.

Diagram



$T_{9,7}$ and one of its graph.